Debt Capacity and Optimal Capital Structure for Privately-Financed Infrastructure Projects

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Abstract: Concession agreements can be used by governments to induce the private sector to develop and operate many types of infrastructure projects. Under this type of arrangement, several private-sector companies join forces, become project promoters, and form a separate company that becomes responsible for financing, building, and operating the facility. Before this company can be formed, prospective promoters must determine how to fund the associated construction and startup costs. They must decide how much to borrow, how much to infuse from their own funds and how much to raise from outside investors. Typically, such projects must repay any debt obligations through their own net operating income, and do not provide the lenders with any other collateral (off-balance-sheet financing). Thus, the possibility of a costly bankruptcy becomes much more likely. In this paper we show that under these circumstances, the amount of debt that a project can accommodate (its debt capacity) is less than 100% debt financing. The amount of debt that maximizes the investors' return on equity is less than the project's debt capacity and the amount of debt that maximizes the project's net present value is even smaller. Exceeding these debt amounts and moving towards debt capacity should be avoided as it can rapidly erode the project's value to the investors. An example illustrates these concepts.

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Introduction

As existing infrastructure ages and demand for new facilities increases due to population growth and technological advancement, governments worldwide no longer have funds in place, or the bonding capacity required, to finance all the public facilities, public services, and infrastructure that they would like to provide. In the US, for example, the availability of federal grants for public works projects has been constrained by budget deficits, while the ability of state and municipal governments to finance construction through bond issues has been affected by changes in tax laws and limits on debt capacity imposed by law, political considerations, or capital markets (Beidleman, 1991). According to Aschauer (1991), the lack of funds to finance infrastructure projects is one of the major causes of the economy's faltering productivity, profitability, and private sector capital formation. He estimated, for example, that a 1% increase in the stock of infrastructure capital would raise American productivity by 0.24%.

Apart from the lack of funding resources, there is an increased understanding on the part of some governments that they should not own and/or operate certain types of facilities and infrastructure because of their less effective utilization of resources, when compared with the more flexible and cost conscious private sector, and because of changes in their political ideologies. Private enterprise can benefit from this situation by providing its financial resources and managerial skills to increase its share of the infrastructure market.

In this paper we describe an arrangement for the private financing of infrastructure projects based on concession agreements. In this context, the objective of the paper is to illustrate how to determine the debt capacity and optimum financial structure for privately-financed infrastructure projects. This decision is of paramount importance because it constrains the ability of the promoting team to go ahead with the project. If the promoting team does not have the necessary equity to achieve the optimal debt-to-equity ratio, then it should search for additional investors until there are enough resources to achieve the optimal capital structure. A promoting team should not try to borrow as much as it can as this would make it worse off. Furthermore, the determination of debt capacity and optimum financial structure provides the basis for the structure and evaluation of the possible types of guarantees (minimum production, minimum revenue, etc.) that the host government may extend to the project (Dias 1994).

To be as realistic as possible in the evaluation of risky debt, the formulation developed here explicitly considers both the possibility of bankruptcy and the effect of taxes. These are the two main factors that influence debt policy (Brealey and Myers 1991). The explicit consideration of bankruptcy costs is of particular importance because privately-financed projects provide no collateral to debtholders. The discussion begins by using a market equilibrium approach to determine the value of the project and of its financial components (debt and equity). Next, we show that when risky debt exists (*i.e.*, when the cost of bankruptcy is greater than zero), there is a limit on the amount that can be borrowed to fund a project (*i.e.*, the project's debt capacity). Finally, we determine the capital structure that maximizes either the investors' return on equity, or the project's *NPV*, and show that these strategies always require debt levels that are less than the project's debt capacity. The application of these concepts is illustrated with an example.

The nature of this important topic requires a relatively complex mathematical treatment. We have chosen to present and explain the most basic analytical results in some detail so that they could be verified. They may also provide a starting point for further investigation by other researchers. Care has been taken to explain most of the results using common-sense concepts so that even if most of the analysis is ignored, the assumptions, results and especially the conclusions can be understood by a wide audience.

Concession Agreements

A concession contract is one possible arrangement governments can use to raise the necessary funds to finance revenue-generating projects when their access to traditional sources of capital is constrained or undesirable. Examples of projects that can be funded using concession arrangements include roads, bridges, tunnels, power plants, pipelines, industrial plants, and office buildings. This type of arrangement requires the involvement of several companies (the promoting team) to finance the project, perform the design, execute and manage its construction, and be responsible for the operation and maintenance of the facility. Depending on the nature of the project, the promoting team might include construction companies, engineering firms, equipment and material suppliers, plant operators, utility companies, and customers of the facility. Figure 1 illustrates possible contractual relationships (dashed lines) and flows of capital (solid lines) among the different participants of a concession-financed project. The shaded boxes indicate those participants that can either be part of the promoting team or serve as external providers of services.

The amount of time promoters have to construct, operate and maintain a facility before transferring its ownership to the project sponsor (usually the government) is known as the concession period. Projects that have finite concession periods are called BOT (Build-Operate-Transfer) projects, otherwise they are called BOO (Build-Operate-Own) projects.

In Build-Operate-Transfer (BOT) projects, the sponsor provides a concession that permits a promoting team to build a facility and to operate it for a specific amount of time. Project promoters use the revenues produced during the concession period to pay back lenders, other shareholders, and to get a return on their investment. After the concession period has elapsed, the operation of the facility and its revenues are transferred to the sponsor that infused, at the time of construction, very few monetary resources. One very well publicized example of this method is the Channel Tunnel project linking France and the UK by rail. Build-Operate-Own (BOO) projects should also produce revenues from their cash flows to cover debt, operation and maintenance costs and to return profit gains to promoter companies. However, project promoters have an unlimited amount of time to operate the facility as well as full ownership of the underlying assets. Actual examples of such projects are power plants (constructed and operated by private utility companies) and public office buildings. This process can be used not only for financing but also for the privatization of public services.

Concession-financed projects are funded through a combination of debt and equity capital. Debt is provided by lending institutions (*e.g.*, banks) while equity is provided by the companies that have an interest in the project (*i.e.*, the promoting companies) and by companies that view the project as an investment opportunity (*e.g.*, pension funds). The use of debt is essential to fund large infrastructure concession projects because promoters rarely have all the necessary financial resources. However, the use of equity is also essential as it complements debt financing and more easily accommodates the financial needs of the project. (Debt instruments present rigid payment dates and amounts and do not normally offer large grace periods. Equity is more flexible as dividends are paid based on the availability of funds.)

Once a project concession is granted by the sponsor, the promoting team creates a company,

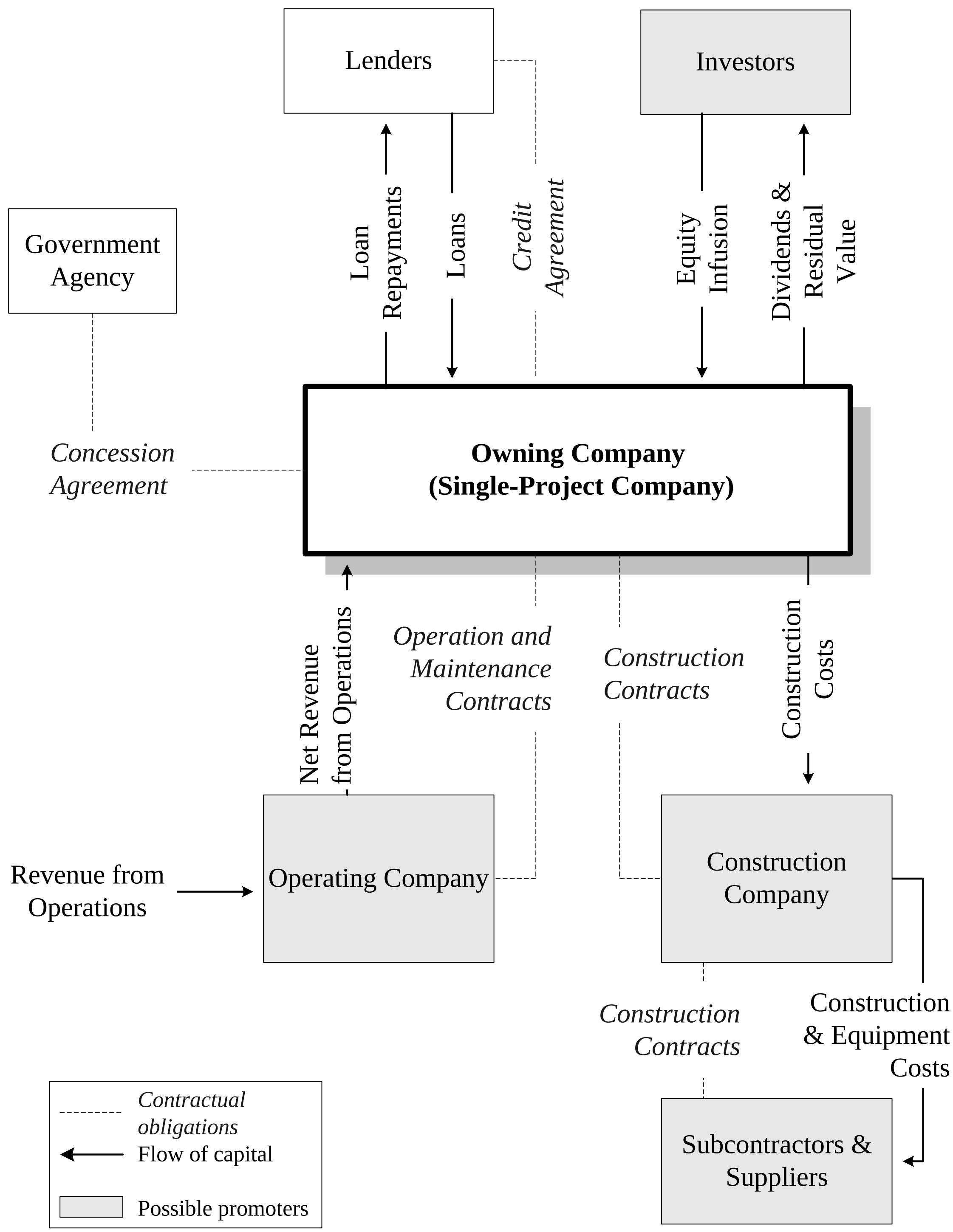

Figure 1: Contractual and Financial Structure of a Privately-Financed Project

referred to as the ‘‘owning company,’’ which is responsible for the financing, construction, and operation of the facility and which retains ownership during the concession period. The creation of an owning company as a separate entity is of great benefit to the promoting companies because it allows them to raise debt without providing a portion of their own assets as collateral. That is, the revenues of the project are the only source to repay the debt. In the case where the project does not produce enough revenues to fully repay the debt, the lenders receive only a partial payment of the debt obligations and do not have any rights to demand full payment from the promoters. This type of financing is known as off-balance-sheet financing. The debt raised to fund the project is not secured by the promoters, and hence it does not appear on their balance-sheets, but only on the balance-sheet of the owning company.

Project Valuation Using the CAPM

No finance theory can give a satisfactory explanation of the valuation of a firm if it fails to take into account the equilibrium of capital markets. The Capital Asset Price Model (CAPM), developed by Sharpe (1964), Lintner (1965), and Mossin (1966) is one such theory. It shows that the equilibrium rate of return on an asset is a function of its relative risk level when compared to the market portfolio. The market portfolio consists of a weighted average of all assets on the market; that is, each asset contributes to the portfolio by the proportion of its value to the total market value of the assets. The essential relationship of the CAPM is:

$$E[\tilde{r}_i] = r_f + \beta_i \left(E[\tilde{r}_m] - r_f\right) \tag{1}$$

Note that throughout this paper random variables are indicated by placing a tilde ($\sim$) over their names. The CAPM indicates that, if the market is in equilibrium at time $t-1$, $E[\tilde{r}_i]$, the expected return on a risky asset (*e.g.*, a project) i during the period $(t-1, t)$, is, r_f, the risk- free rate of interest during that period plus a risk premium, which is determined by $E[\tilde{r}_m] - r_f$, the excess rate of return on the market portfolio (above the risk-free rate) and β_i, the systematic risk of asset i. Systematic risk, also called market risk, exists because there are economy-wide factors that affect the entire market and cannot be avoided no matter how much diversified a portfolio of assets is. It is measured by determining the sensitivity of the returns on asset i to market movements, that is:

$$\beta_i = \frac{\text{Cov}(\tilde{r}_i, \tilde{r}_m)}{\sigma_m^2} \tag{2}$$

An asset that has $\beta > 1$ is more sensitive to market movements than the market portfolio, and thus more risky, and should provide returns greater than the expected return on the market portfolio. Similarly, an asset with $\beta < 1$ is less risky than the market portfolio. The derivation of the CAPM, as well as its underlying assumptions, appear in Copeland (1988, pp.195-198).

Hamada (1971) notes that, in a single-period situation, the CAPM relationship (Eq. 1) can be viewed not only as the market equilibrium relationship between the expected rate of return on asset i and its individual risk, but also as a minimum expected rate of return required by the market for a given level of systematic risk. Thus, it provides a cut-off rate against which the expected rate of return on project i can be compared. For example, if the cost of investing in project i is A_i, its expected value at the end-of-period is $E[\tilde{V}_{i,1}]$, and its systematic risk is β_i then, in order to be accepted (*i.e.*, to have a positive net present value) the project must satisfy the following condition:

$$\frac{E[\tilde{V}_{i,1}]}{1 + E[\tilde{r}_i]} = \frac{E[\tilde{V}_{i,1}]}{1 + r_f + \beta_i \left(E[\tilde{r}_m] - r_f\right)} \geq A_i \tag{3}$$

Based on Hamada's interpretation, the CAPM can be used to determine the present value of a project when the market is in equilibrium. To see this let us define the following rates of return:

$$\tilde{R} = 1 + \tilde{r} \text{ (one plus the rate of return on a single-period project),} \tag{4}$$

$$\tilde{R}_m = 1 + \tilde{r}_m \text{ (one plus the rate of return on the market), and} \tag{5}$$

$$R_f = 1 + r_f \text{ (one plus the risk-free rate of interest)} \tag{6}$$

Note that throughout this paper we show variables that represent one-plus-the-rate-of-return with capital letters (*e.g.*$\tilde{R}_i = 1 + \tilde{r}_i$ and $\widetilde{ROE} = 1 + \widetilde{Roe}$). Given this convention, (1) can be expressed as:

$$E[\tilde{R}] = R_f + \frac{\text{Cov}(\tilde{R}, \tilde{R}_m)}{\sigma_m^2} \left(E[\tilde{R}_m] - R_f\right) \tag{7}$$

By definition, we have:

$$\tilde{R} = \frac{\tilde{V}_1}{V} \tag{8}$$

$$\text{Cov}(\tilde{R}, \tilde{R}_m) = \text{Cov}(\frac{\tilde{V}_1}{V}, \tilde{R}_m) = \frac{1}{V}\ \text{Cov}(\tilde{V}_1, \tilde{R}_m) \tag{9}$$

where V is the present (actual) market value of the project when the market is in equilibrium and $\tilde{V}_1$ is the uncertain end-of-period value of the project. Substituting (8) and (9) into (7) gives:

$$\frac{E[\tilde{V}_1]}{V} = R_f + \left(E[\tilde{R}_m] - R_f\right) \frac{\text{Cov}(\tilde{V}_1, \tilde{R}_m)}{\sigma_m^2 V} \tag{10}$$

and rearranging the terms:

$$V = \frac{E[\tilde{V}_1] - \frac{E[\tilde{R}_m] - R_f}{\sigma_m^2} \text{Cov}(\tilde{V}_1, \tilde{R}_m)}{R_f} = \frac{E[\tilde{V}_1] - \lambda \text{ Cov}(\tilde{V}_1, \tilde{R}_m)}{R_f} \tag{11}$$

where λ is the market price of a unit of risk. Note that $E[\tilde{V}_1] - \lambda \text{ Cov}(\tilde{V}_1, \tilde{R}_m)$ is the certainty equivalent (as determined by the market) of the end-of-period value of project $\tilde{V}_1$, and that is why it is discounted by the risk-free rate (instead of $\tilde{R}$) in order to calculate the actual market value of the project.

Bankruptcy Costs

Let us consider a one-period privately-financed project, that costs a certain amount A to be built, is financed through the use of equity and debt, and generates a net operating income $\tilde{X}$ at the end of its operational period. Then, the end-of-period market value of the project, $\tilde{V}_1$, can be calculated by summing the end-of-period market values of the outstanding debt and equity:

$$\tilde{V}_1 = \tilde{D}_1 + \tilde{S}_1 \tag{12}$$

The end-of-period market value of equity, $\tilde{S}_1$, is uncertain as the earnings received by the equityholders depend on the net operating income of the project, $\tilde{X}$, and on the the amount of debt outstanding. The end-of-period market value of debt, $\tilde{D}_1$, is uncertain because it also depends on the net operating income of the project and because the debt repayment is not guaranteed by the promoting companies. If the net operating income, $\tilde{X}$, is greater than the amount borrowed at the beginning of the project (debt principal) plus the promised interest, then the debtholders will receive the full promised amount d_1 (principal plus interest) at the end of the period. On the other hand, if the project does not produce a net operating income sufficient to repay the debt ($\tilde{X} < d_1$), the owning company does not meet its debt obligation, enters a state of financial distress and becomes bankrupt. In this case, debtholders take ownership of the company and pay the costs of bankruptcy before they receive any payment. Details of this process are described in Martin and Scott (1976), Hong and Rappaport (1978), and Kim (1978).

Kim (1978) discusses the different types of bankruptcy costs and classifies them into two categories: direct costs and indirect costs. For infrastructure projects, direct costs include administrative expenses (*e.g.*, legal fees, trustee fees, referee fees, and time lost by executives in litigation). Indirect costs are incurred basically in the form of trustee certificates. These certificates are used to raise new capital for the continuance of the services provided by the project facility and become senior instruments to the outstanding debt of the bankrupt company. In this paper, bankruptcy costs are represented by the following linear function (Kim 1978):

$$\tilde{B} = b_f + b_v \tilde{X} \quad (0 \leq \tilde{B} \leq \tilde{X}) \tag{13}$$

where $\tilde{B}$ represents the uncertain cost of bankruptcy and is a positive-non-greater-than function of $\tilde{X}$; b_f represents the expected value of the components of bankruptcy costs (expressed in monetary units) that are independent of the company's net operating income $\tilde{X}$ (*i.e.*, those costs that do not depend on the size of the owning company); and b_v is a variable cost coefficient that can assume values from -1 to 1 and which relates the costs of bankruptcy, $\tilde{B}$, to the net operating income $\tilde{X}$ (if $\tilde{B}$ is independent of $\tilde{X}$ then b_v is zero).

For convenience, it is assumed that once in bankruptcy the owning company is liquidated and its proceedings get distributed according to the Bankruptcy Reform Act of 1978. Thus, administrative expenses associated with liquidating the project (*i.e.*, bankruptcy costs), such as fees and other compensation paid to trustees, attorneys, accountants, etc., are paid before the debtholders claims on the project assets. Empirical studies on bankruptcies show that administrative expenses range from 4 to 20% of a company's assets depending on the type of company analyzed and other sample characteristics (Van Horne, 1986).

The Present Value of Debt

For a one-period privately-financed project, the amount D that the owning company can borrow depends on the risk characteristics of the amount d_1 it promises to repay at the end of the period,

$$d_1 = D(1 + \mathit{Int}) \tag{14}$$

where *Int* is the nominal interest rate charged by the lenders. Because the loan is risky, however, the amount $E[\tilde{D}_1]$ that the lenders expect to receive is less than the full promised amount d_1 and their expected return $E[\tilde{r}_D]$ is less than *Int*:

$$E[\tilde{D}_1] = D(1 + E[\tilde{r}_D]) \tag{15}$$

For the same reason, as the promised amount d_1 increases, so does the risk faced by the debtholders and so does the the nominal interest rate *Int* they demand. As shown below, when d_1 reaches a certain level, the required nominal interest *Int* is so large that the debt amount D can actually decrease (even though the owning company promises to pay more). To illustrate this behavior in a mathematically tractable manner, and without loss of generality, the remaining discussion focuses on the evaluation of debt and equity for a one-period project. The analysis for multiperiod projects, although similar, is best undertaken using numerical methods.

The loan amount D (*i.e.*, the present value of a project's debt as determined by the market) can be computed by following the same line of reasoning used to determine V, the present market value of a project (Eq. 11),

$$D = \frac{E[\tilde{D}_1] - \lambda \, \text{Cov}(\tilde{D}_1, \tilde{R}_m)}{R_f} \tag{16}$$

where $E[\tilde{D}_1]$ is the expected value of the debt at time 1; λ is the market price per unit of risk; $\text{Cov}(\tilde{D}_1, \tilde{R}_m)$ is the covariance between the value of debt at time 1 and one-plus-the-rate-of-return-on-the-market ; and R_f is one-plus-the-risk-free-rate.

The end-of-period value of debt, $\tilde{D}_1$, depends on the end-of-period project net operating income, $\tilde{X}$, and can be expressed as:

$$\tilde{D}_1 = \begin{cases} d_1 & \text{if } \tilde{X} \geq d_1 \\ \tilde{X} - \tilde{B} & \text{if } \tilde{B} \leq \tilde{X} < d_1 \\ 0 & \text{if } \tilde{X} < \tilde{B}, \textit{ i.e.}, \tilde{X} < b' = \frac{b_f}{1-b_v} \end{cases} \tag{17}$$

Thus, if the net operating income at the end of the period is greater than the promised amount d_1, the debtholders receive the full debt payments. Otherwise, they receive the net operating income minus the bankruptcy costs, provided this difference is positive, and nothing if the difference is negative (the entire net operating income is consumed by bankruptcy costs). Alternatively, $\tilde{D}_1$ can be expressed in the following equation form:

$$\tilde{D}_1 = d_1(1 - \delta_b)\delta_q + \delta_b\delta_q\tilde{X} - \delta_b\delta_q(b_f + b_v\tilde{X}) \tag{18}$$

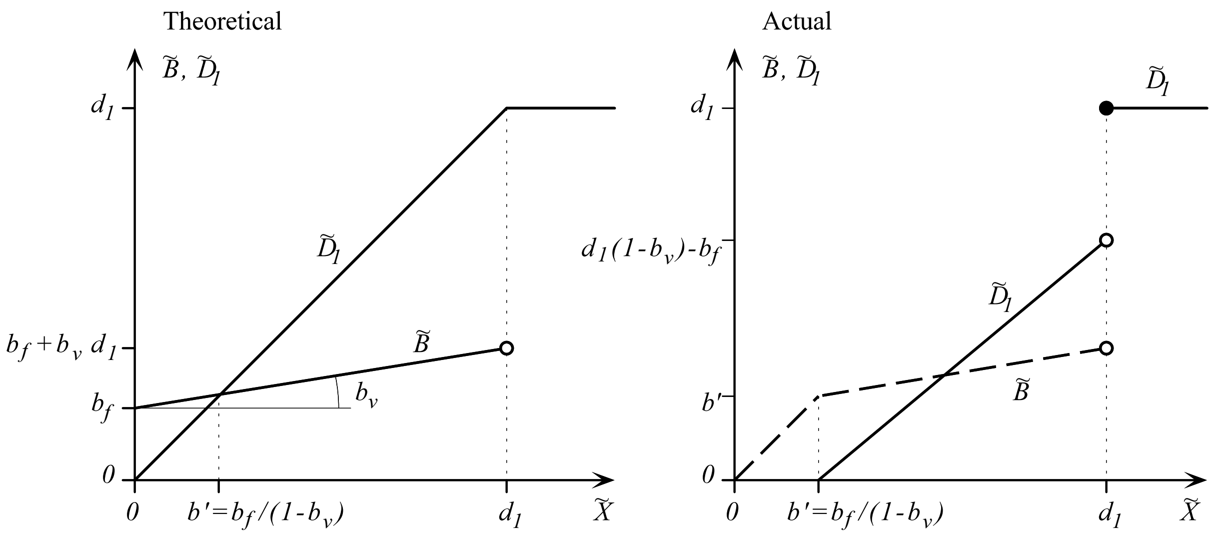

Figure 2: Value of Debt ($\tilde{D}_1$) and Bankruptcy Costs ($\tilde{B}$) as a Function of the NOI ($\tilde{X}$)

where δ_b and δ_q are binary variables defined as follows:

$$\delta_b = \begin{cases} 0 & \text{if } \tilde{X} \geq d_1 \\ 1 & \text{if } \tilde{X} < d_1 \end{cases} \tag{19}$$

$$\delta_q = \begin{cases} 0 & \text{if } \tilde{X} < b' = \frac{b_f}{1-b_v} \\ 1 & \text{if } \tilde{X} \geq b' \end{cases} \tag{20}$$

Note that when the owning company is not bankrupt, ***i.e.***, $\tilde{X} \geq d_1$, then $\delta_b = 0$, $\delta_q = 1$, and $\tilde{D}_1 = d_1$. Similarly, when in bankruptcy and $\tilde{X} \geq b'$, ***i.e.***, $\tilde{B} \leq \tilde{X} < d_1$, then $\delta_b = 1$, $\delta_q = 1$, and $\tilde{D}_1 = (1 - b_v)\tilde{X} - b_f$. Finally, when in bankruptcy and $\tilde{X} < b'$, ***i.e.***, $\tilde{X} < \tilde{B}$, then $\delta_b = 1$, $\delta_q = 0$, and $\tilde{D}_1 = 0$. Figure 2 shows the value of $\tilde{D}_1$ as a function of $\tilde{X}$. The graph on the left, shows $\tilde{D}_1$ (when bankruptcy costs are not considered) and $\tilde{B}$ (without the restriction $\tilde{B} \leq \tilde{X}$). The graph on the right, illustrates the actual values of $\tilde{D}_1$ given by (18), when bankruptcy costs are considered, and the values of $\tilde{B}$ with the restriction $\tilde{B} \leq \tilde{X}$ imposed.

The expected value of the end-of-period debt, $E[\tilde{D}_1]$, can be calculated from (18) as:

$$E[\tilde{D}_1] = d_1 \left(E[\delta_q] - E[\delta_b \delta_q]\right) + (1 - b_v)E[\delta_b \delta_q \tilde{X}] - b_f E[\delta_b \delta_q] \tag{21}$$

The expected values on the right-hand side of (21) are derived in Appendix II under the assumption that $\tilde{X}$ follows a Normal distribution. Substituting (65), (66), and (68) into (21) and rearranging the terms gives:

$$\begin{aligned} E[\tilde{D}_1] &= d_1\,(1 - F_X(d_1)) + (1 - b_v)\left\{E[\tilde{X}]\,(F_X(d_1) - F_X(b')) + \sigma_X^2\,(f_X(b') - f_X(d_1))\right\} \\ &\quad - b_f\,(F_X(d_1) - F_X(b')) \end{aligned} \tag{22}$$

Therefore, the expected end-of-period payment to debtholders after bankruptcy costs (*i.e.*, the value of debt at time 1) is the full promised amount d_1 multiplied by the probability that the project does not go bankrupt plus the conditional expected end-of-period project net cash flow given that the project is bankrupt minus the expected value of the bankruptcy costs.

The $\text{Cov}(\tilde{D}_1, \tilde{R}_m)$ in (16) can be expressed as:

$$\begin{aligned} \text{Cov}(\tilde{D}_1, \tilde{R}_m) &= \text{Cov}(d_1(1-\delta_b)\delta_q + \delta_b\delta_q\tilde{X} - \delta_b\delta_q(b_f + b_v\tilde{X}), \tilde{R}_m) \\ &= d_1\ \text{Cov}(\delta_q, \tilde{R}_m) - (d_1 + b_f)\ \text{Cov}(\delta_b\delta_q, \tilde{R}_m) + (1 - b_v)\ \text{Cov}(\delta_b\delta_q\tilde{X}, \tilde{R}_m) \end{aligned} \tag{23}$$

Substituting the covariances on the right-hand side of the above equation by (74) and (76) (Appendix II), rearranging terms, and multiplying both sides by λ gives:

$$\lambda\ \text{Cov}(\tilde{D}_1, \tilde{R}_m) = \{(1 - b_v)\,(F_X(d_1) - F_X(b')) + (b_f + b_v d_1) f_X(d_1)\}\ \lambda\ \text{Cov}(\tilde{X}, \tilde{R}_m) \tag{24}$$

Eq. 24 shows that the systematic risk premium on the project's debt, $\lambda\ \text{Cov}(\tilde{D}_1, \tilde{R}_m)$, is equal to the project's systematic operating risk premium, $\lambda\ \text{Cov}(\tilde{X}, \tilde{R}_m)$, multiplied by a factor that represents the probability that debtholders would only receive some partial payment (given by the occurrence of bankruptcy) plus the systematic risk premium on the project's bankruptcy costs, $\lambda\ \text{Cov}(\tilde{X}, \tilde{R}_m)[(b_f + b_v d_1) f_X(d_1) - b_v(F_X(d_1) - F_X(b'))]$.

Therefore, if the company is not bankrupt at the end of the period (or if the company has a 0% probability of going into bankruptcy), debtholders receive a fixed amount d_1 that has no systematic relationship with the market. It is only when the company is bankrupt that $\tilde{D}_1$ (and also $\tilde{X}$) presents a systematic risk that cannot be diversified away by the debtholders. This is reflected in the following equation for β_D:

$$\begin{aligned} \beta_D &= \frac{\text{Cov}(\tilde{R}_D, \tilde{R}_m)}{\sigma_m^2} = \frac{\text{Cov}(\tilde{D}_1, \tilde{R}_m)}{D\sigma_m^2} \\ &= \frac{\text{Cov}(\tilde{X}, \tilde{R}_m)}{D\sigma_m^2}\,\{(1 - b_v)\,(F_X(d_1) - F_X(b')) + (b_f + b_v d_1) f_X(d_1)\} \end{aligned} \tag{25}$$

From (25) one can see that β_D increases as bankruptcy costs increase. This can be shown by examining the terms in (25) that depend on b_f and b_v. Given that $d_1 \leq E[\tilde{X}]$, the area of a rectangle with base d_1 and height $f_X(d_1)$ is always greater than the area under a Normal distribution $F_X(d_1) - F_X(b')$. Thus, it follows that $(b_f + b_v d_1) f_X(d_1)$ is always greater than $b_v(F_X(d_1) - F_X(b'))$. This relationship together with the condition $\text{Cov}(\tilde{X}, \tilde{R}_m) > 0$ implies that higher bankruptcy costs result in a higher systematic risk premium on the project's debt. In other words, as bankruptcy costs increase so does the risk premium required by the debtholders.

The market value of debt, D, can be calculated by substituting (22) and (24) into (16):

$$\begin{aligned} D \;=\; & \frac{1}{R_f}\Big\{d_1\left(1 - F_X(d_1)\right) + (1 - b_v)\Big\{E[\tilde{X}]\left(F_X(d_1) - F_X(b')\right) + \sigma_X^2\left(f_X(b') - f_X(d_1)\right)\Big\} - \\ & -b_f\left(F_X(d_1) - F_X(b')\right) - \lambda\,\text{Cov}(\tilde{X}, \tilde{R}_m)\left[(1 - b_v)\left(F_X(d_1) - F_X(b')\right) + \right. \\ & \left. +(b_f + b_v d_1) f_X(d_1)\right]\Big\} \end{aligned} \tag{26}$$

Note that, in the absence of bankruptcy costs (***i.e.***, $b_f = b_v = 0$), the present value of debt (***i.e.***, the amount that the owning company can borrow) is the promised amount d_1 to be paid to debtholders at the end of the period multiplied by the probability that the company does not go bankrupt plus the conditional expected value of the project's net income given that the company goes into bankruptcy minus the company's operating risk premium, $\lambda\,\text{Cov}(\tilde{X}, \tilde{R}_m)$, multiplied by the company's probability of bankruptcy (all discounted at the risk-free rate).

The Present Value of Equity

The amount of money that must be invested as equity in a project that costs A, given a debt level D, is $A - D$. As soon as this money is invested in a project with a positive *NPV*, however, its market value increases to S, and the net present value of the project is

$$NPV = S - (A - D) = S + D - A = V - A \tag{27}$$

The present (market) value of S can be expressed in the same form used to express the actual market value of the debt of a project, D, that is

$$S = \frac{E[\tilde{S}_1] - \lambda\,\text{Cov}(\tilde{S}_1, \tilde{R}_m)}{R_f} \tag{28}$$

where $E[\tilde{S}_1]$ is the expected value of the end-of-period value of equity and $\text{Cov}(\tilde{S}_1, \tilde{R}_m)$ is the covariance between the end-of- period value of equity and the return on the market.

The end-of-period value of equity, $\tilde{S}_1$, represents the market value of the company after debt obligations are paid to debtholders and taxes are paid to the government, that is:

$$\tilde{S}_1 = (1-T)(\tilde{X} - D\mathit{Int} - \mathit{Dep}) + \mathit{Dep} - D \quad (29)$$

where $D\mathit{Int}$ is the interest due on the debt and *Dep* is depreciation. As $D(1+\mathit{Int}) = d_1$ and assuming that $\mathit{Dep} = A$, (29) can be rewritten as:

$$\tilde{S}_1 = (1-T)(\tilde{X} - d_1) + T(A - D) \quad (30)$$

Equityholders have limited liability and do not have any obligations to pay if the company goes bankrupt (*i.e.*, if $\tilde{X} < d_1$ then $\tilde{S}_1 = 0$). As a result, $\tilde{S}_1$ can be expressed more accurately as:

$$\tilde{S}_1 = \begin{cases} (1-T)(\tilde{X} - d_1) + T(A - D) & \text{if } \tilde{X} \geq d_1 \\ 0 & \text{if } \tilde{X} < d_1 \end{cases} \quad (31)$$

Alternatively, $\tilde{S}_1$ can be expressed in the following equation form:

$$\tilde{S}_1 = (1-T)[\tilde{X} - d_1](1-\delta_b) + T(A-D)(1-\delta_b) \quad (32)$$

Using the relationships developed in Appendix II, the expected value of the end-of-period equity, $E[\tilde{S}_1]$, is:

$$E[\tilde{S}_1] = \Big[(1-T)(E[\tilde{X}] - d_1) + T(A-D)\Big](1 - F_X(d_1)) + (1-T)\sigma_X^2 f_X(d_1) \quad (33)$$

Therefore, the expected end-of-period value of the owning company after all obligations have been satisfied is the after-tax conditional expected value of the project's net operating income given that the company is not bankrupt, $(1-T)[E[\tilde{X}](1-F_X(d_1)) + \sigma_X^2 f_X(d_1)]$, minus the after-tax value of the debt obligations multiplied by the probability that the company is not bankrupt, $(1-T)d_1(1-F_X(d_1))$, plus the expected value of the tax credits, $T(A-D)(1-F_X(d_1))$.

Following the procedure used to calculate $\text{Cov}(\tilde{D}_1, \tilde{R}_m)$, the systematic risk premium on the company's equity, $\lambda\ \text{Cov}(\tilde{S}_1, \tilde{R}_m)$, is given by:

$$\lambda\ \text{Cov}(\tilde{S}_1, \tilde{R}_m) = \{(1-T)(1-F_X(d_1)) + T(A-D)f_X(d_1)\}\ \lambda\ \text{Cov}(\tilde{X}, \tilde{R}_m) \quad (34)$$

Here, $\lambda\ \text{Cov}(\tilde{S}_1, \tilde{R}_m)$ is equal to the after-tax project's systematic operating risk premium, $(1-T)\lambda\ \text{Cov}(\tilde{X}, \tilde{R}_m)$, multiplied by the probability that the company does not go bankrupt plus the systematic risk premium on tax credits, $T(A-D)f_X(d_1)\lambda\ \text{Cov}(\tilde{X}, \tilde{R}_m)$.

Finally, the present (market) value of equity, S, can be calculated by substituting (33) and (34) into (28):

$$\begin{aligned} S &= \frac{1}{R_f}\left\{\left[(1-T)(E[\tilde{X}]-d_1)+T(A-D)\right](1-F_X(d_1))+(1-T)\sigma_X^2 f_X(d_1)\right. \\ &\quad \left. -\lambda\ \text{Cov}(\tilde{X},\tilde{R}_m)\left[(1-T)(1-F_X(d_1))+T(A-D)f_X(d_1)\right]\right\} \end{aligned} \tag{35}$$

Project Debt Capacity

Project debt capacity, D^c, is defined as the maximum amount an owning company can borrow in a perfect capital market in order to fund a project. For the concept of debt capacity to be meaningful, it is necessary to show that there exists a finite value d_1^c, that satisfies the following two conditions: $\partial D/\partial d_1 = 0$ and $\partial^2 D/\partial d_1^2 < 0$. The first derivative is given by differentiating equation (26) with respect to d_1:

$$\begin{aligned} \frac{\partial D}{\partial d_1} &= \frac{1}{R_f}\left\{\frac{\partial d_1}{\partial d_1}-\frac{\partial\left[d_1 F_X(d_1)\right]}{\partial d_1}+(1-b_v)\left[E[\tilde{X}]\frac{\partial F_X(d_1)}{\partial d_1}-\sigma_X^2\frac{\partial f_X(d_1)}{\partial d_1}\right]-\right. \\ &\quad \left. -b_f\frac{\partial F_X(d_1)}{\partial d_1}-\lambda\ \text{Cov}(\tilde{X},\tilde{R}_m)\left[(1-b_v)\frac{\partial F_X(d_1)}{\partial d_1}+b_f\frac{\partial f_X(d_1)}{\partial d_1}+b_v\frac{\partial\left[d_1 f_X(d_1)\right]}{\partial d_1}\right]\right\} \end{aligned} \tag{36}$$

After performing differentiations, substituting $\frac{\partial F_X(d_1)}{\partial d_1}$ by $f_X(d_1)$ and $\frac{\partial f_X(d_1)}{\partial d_1}$ by $-(\frac{d_1-E[\tilde{X}]}{\sigma_X^2})f_X(d_1)$, and collecting terms, (36) becomes:

$$\begin{aligned} \frac{\partial D}{\partial d_1} &= \frac{1}{R_f}\left\{1-F_X(d_1)-(b_f+b_v d_1)f_X(d_1)-\right. \\ &\quad \left. -\lambda\ \text{Cov}(\tilde{X},\tilde{R}_m)f_X(d_1)\left[1-(\frac{d_1-E[\tilde{X}]}{\sigma_X^2})(b_f+b_v d_1)\right]\right\} \end{aligned} \tag{37}$$

Setting (37) equal to zero gives:

$$\begin{aligned} 1-F_X(d_1) &= (b_f+b_v d_1)f_X(d_1)+ \\ &\quad +\lambda\ \text{Cov}(\tilde{X},\tilde{R}_m)f_X(d_1)\left[1-(\frac{d_1-E[\tilde{X}]}{\sigma_X^2})(b_f+b_v d_1)\right] \end{aligned} \tag{38}$$

The second derivative, $\partial^2 D/\partial d_1^2$, can be calculated by differentiating (37) with respect to d_1:

$$\begin{aligned} \frac{\partial^2 D}{\partial d_1^2} &= \frac{f_X(d_1)}{R_f}\left\{-(1+b_v)+\left(\frac{d_1-E[\tilde{X}]}{\sigma_X^2}\right)\left[(b_f+b_v d_1)+\right.\right. \\ &\quad \left.\left. +\lambda\ \text{Cov}(\tilde{X},\tilde{R}_m)\left(1-(b_f+b_v d_1)\left(\frac{d_1-E[\tilde{X}]}{\sigma_X^2}-b_v\right)\right)\right]\right\} \end{aligned} \tag{39}$$

An inspection of (39) shows that $\partial^2 D/\partial d_1^2 < 0$ for any $d_1 < E[\tilde{X}]$. Hence, as long as the first condition is met inside the interval $0 \le D < A$, d_1^c corresponds to a maximum.

According to Rolle's theorem, d_1^c exists only when the right-hand side (RHS) of (38) can assume values greater than the left-hand side (LHS). This is because at low values of d_1, (LHS) $>$ (RHS) and $\partial D/\partial d_1 > 0$. Thus, values of d_1 that satisfy (LHS) $<$ (RHS) inside the interval $0 \le D < A$, (or within $0 \le d_1 < A(1 + \mathit{Int})$) assure that $\partial D/\partial d_1 = 0$ at some finite point d_1^c. Therefore, D^c can be calculated by substituting the promised amount d_1^c that satisfies (38) into (16).

Eq. 38 can be arranged to show how the level of bankruptcy costs affects the existence of the debt capacity of an owning company:

$$b_f + b_v d_1 \quad > \quad \frac{1 - F_X(d_1) - \lambda \,\text{Cov}(\tilde{X}, \tilde{R}_m) f_X(d_1)}{f_X(d_1)\left[1 - (\frac{d_1 - E[\tilde{X}]}{\sigma_X^2})\lambda \,\text{Cov}(\tilde{X}, \tilde{R}_m)\right]} \tag{40}$$

Thus, if the bankruptcy costs satisfy the above condition and $0 \le d_1 < A(1 + \mathit{Int})$ then there is a finite limit on the owning company's debt capacity. If bankruptcy is costless, but there is still the possibility the company might go bankrupt, debt capacity exists as long as $\lambda \,\text{Cov}(\tilde{X}, \tilde{R}_m) > \frac{1 - F_X(d_1)}{f_X(d_1)}$.

Notice that the numerator in (40), $1 - F_X(d_1) - \lambda \,\text{Cov}(\tilde{X}, \tilde{R}_m) f_X(d_1)$, is the same as the certainty equivalent of one dollar associated with the occurrence of bankruptcy, $E[\delta_b] - \lambda \,\text{Cov}(\delta_b, \tilde{R}_m)$ (see (64) and (73)). Thus, in the extreme where bankruptcy becomes certain, the numerator in (40) becomes zero and from (38) we see that RHS $>$ LHS. Consequently, at the same extreme, $\partial D/\partial d_1$ is reduced to:

$$\frac{\partial D}{\partial d_1} \quad = \quad \frac{f_X(d_1)}{R_f}\left\{(1 - b_v)(d_1 - E[\tilde{X}]) - \lambda \,\text{Cov}(\tilde{X}, \tilde{R}_m)\left(-b_f \frac{d_1 - E[\tilde{X}]}{\sigma_X^2} + b_v\right)\right\} \tag{41}$$

The above equation illustrates that, when bankruptcy is certain and costly, $\partial D/\partial d_1$ is always negative. Hence, **debt capacity is always reached (*i.e.,* $\partial D/\partial d_1 = 0$) before bankruptcy becomes certain. This means, that in the presence of bankruptcy costs that satisfy (40) within the interval $0 \le d_1 < A(1 + \mathit{Int})$, the owning company can never borrow 100% of the project's costs even if it wants to. Promising to pay more in the future (*i.e.,* increasing d_1 beyond d_1^c) does not increase D because of the higher risk of bankruptcy and its associated costs.**

Optimal Capital Structure

The optimal financial structure of an owning company is defined here as the combination of debt and equity that achieves a financial objective. Two such objectives are investigated here: maximizing the return on the equityholders' investment (*ROE*) and maximizing the project's net present value (*NPV*).

The *ROE* is calculated by dividing the end-of-period value of the project after all obligations have been paid (*i.e.*, expenses, debt and taxes) by the amount initially infused by project investors, that is:

$$\widetilde{ROE} = \frac{\tilde{S}_1}{A - D} \tag{42}$$

In order to determine the financial structure that maximizes the return to project investors it is necessary to follow a procedure similar to the one used to determine debt capacity, that is, to set $\partial E[\widetilde{ROE}]/\partial d_1 = 0$ and to verify that $\partial^2 E[\widetilde{ROE}]/\partial d_1^2 < 0$.

Differentiating $\frac{E[\tilde{S}_1]}{A-D}$ with respect to d_1 gives:

$$\frac{\partial E[\widetilde{ROE}]}{\partial d_1} = \frac{\partial}{\partial d_1}\left(\frac{E[\tilde{S}_1]}{(A-D)}\right) = \frac{\frac{\partial E[\tilde{S}_1]}{\partial d_1}(A-D) + E[\tilde{S}_1]\frac{\partial D}{\partial d_1}}{(A-D)^2} \tag{43}$$

$$= \frac{1}{(A-D)^2}\Bigg\{(A-D)\Bigg[-(1-T)(1-F_X(d_1)) - T\left((A-D)f_X(d_1) + \frac{\partial D}{\partial d_1}(1-F_X(d_1))\right)\Bigg] + E[\tilde{S}_1]\frac{\partial D}{\partial d_1}\Bigg\} \tag{44}$$

As the optimal capital structure occurs when $\frac{\partial E[\widetilde{ROE}]}{\partial d_1} = 0$, (44) yields:

$$E[\tilde{S}_1]\frac{\partial D}{\partial d_1} = (A-D)\left\{(1-T)(1-F_X(d_1)) + T\left[(A-D)f_X(d_1) + \frac{\partial D}{\partial d_1}(1-F_X(d_1))\right]\right\} \tag{45}$$

and solving for $\frac{\partial D}{\partial d_1}$ gives:

$$\frac{\partial D}{\partial d_1} = \frac{(1-T)(1-F_X(d_1)) + T(A-D)f_X(d_1)}{\frac{E[\tilde{S}_1]}{A-D} - T(1-F_X(d_1))} \tag{46}$$

Note that the numerator and the denominator of (46) are positive for $d_1 < d_1^c$ and $A > D$. (In order to prove that the denominator of (46) is always positive, it is only necessary to show that $E[\tilde{S}_1] > T(A-D)(1-F_X(d_1))$ which is trivial because $0 < T(1-F_X(d_1)) < 1$ and

$E[\tilde{S}_1] \geq A - D$ for a positive *NPV* project.) Consequently, when $\partial E[\widetilde{ROE}]/\partial d_1 = 0$ we always have $\partial D/\partial d_1 > 0$. Thus, the company's optimal capital structure always occurs before its debt capacity is reached, $d_1^{ROE} < d_1^c$, where d_1^{ROE}, the promised debt amount that maximizes the return to equityholders, is the value of d_1 that satisfies (46). Therefore, **when debt capacity does not allow 100% debt financing ($A > D$), an owning company that wants to maximize its return on investment should borrow at less than debt capacity.** If the project's debt capacity allows 100% debt financing (*i.e.*, $D = A$), (46) gives $\partial D/\partial d_1 = 0$ and the optimal capital structure occurs at 100% debt financing.

A similar analysis can be undertaken for the objective of maximizing the project's net present value. From (27) we see that the optimal capital structure that maximizes *NPV* is exactly the same as the amount of debt and equity that maximizes the wealth of the equityholder in traditional finance (Brealey and Myers 1991), that is,

$$\frac{\partial NPV}{\partial d_1} = \frac{\partial V}{\partial d_1} = \frac{\partial D}{\partial d_1} + \frac{\partial S}{\partial d_1} = 0 \tag{47}$$

The objective of maximizing the equityholders' wealth does not usually provide the same ''optimal'' capital structure as the objective of maximizing their returns. The two objectives provide similar results only when

$$\left.\frac{\partial V}{\partial d_1}\right|_{d_1^{ROE}} = \left.\frac{\partial E[\widetilde{ROE}]}{\partial d_1}\right|_{d_1^{ROE}} = 0 \tag{48}$$

and this implies:

$$E[\widetilde{ROE} \mid d_1 = d_1^{ROE}] \approx R_f \tag{49}$$

In order to see this, substitute $\frac{\partial D}{\partial d_1}$ by $-\frac{\partial S}{\partial d_1}$ (from (47)), $E[\tilde{S}_1] = E[\widetilde{ROE}](A - D)$ (from (42)) into (43) and let d_1^V be the value of d_1 that satisfies (47),

$$\left.\frac{\partial E[\widetilde{ROE}]}{\partial d_1}\right|_{d_1^V} = \frac{1}{A - D}\left.\left[\frac{\partial E[\tilde{S}_1]}{\partial d_1} - E[\widetilde{ROE}]\frac{\partial S}{\partial d_1}\right]\right|_{d_1^V} \tag{50}$$

Setting the right-hand side of (50) equal to zero gives:

$$E[\widetilde{ROE}] = \frac{\partial E[\tilde{S}_1]/\partial d_1}{\partial S/\partial d_1} \tag{51}$$

Substituting $\frac{\partial E[\tilde{S}_1]}{\partial d_1}$ by the definition given in (28) yields:

$$E[\widetilde{ROE}] = Rf + \lambda \frac{\frac{\partial}{\partial d_1}\ \text{Cov}(\tilde{S}_1, \tilde{R}_m)}{\frac{\partial S}{\partial d_1}} \approx R_f \tag{52}$$

Table 1: Input Parameters for the Example Project (all $ values in millions)

Project		Market	
Variable (1)	Value (2)	Variable (3)	Value (4)
A	$ 2,170	$E[\tilde{R}_m]$	1.14
$E[\tilde{X}]$	$ 2,750	σ_m	0.25
σ_X	$ 800	T	0.35
ρ_{X,R_m}	0.70	R_f	1.06
b_f	$ 100		
b_v	0.30		

The above derivation implies that if $d_1^V \neq d_1^{ROE}$ then the objective of maximizing the $E[\widetilde{ROE}]$ does not provide an ''optimal capital structure'' similar to the objective of maximizing stockholders' wealth. More specifically, $d_1^V < d_1^{ROE}$ if:

$$E[\widetilde{ROE}]\Big|_{d_1^{ROE}} > Rf + \lambda \frac{\frac{\partial}{\partial d_1}\ \text{Cov}(\tilde{S}_1, \tilde{R}_m)}{\frac{\partial S}{\partial d_1}} \approx R_f \tag{53}$$

In other words, since (53) should always be true, **the maximization of return on equity investment always allows more borrowing than the maximization of the company's net present value.** This is made evident by the following example.

Example

This section presents an example to illustrate the concepts developed in previous sections. Table 1 shows the input parameters necessary for the determination of the debt capacity and the optimal capital structure of a privately-financed project and displays the specific values assumed for the parameters in this example.

Table 2 contains the numerical values of D, S, V, *NPV*, D/A, $E[\tilde{r}_D]$, *Int*, $E[\tilde{r}_S]$, $E[\widetilde{Roe}]$, $\partial D/\partial d_1$, $\partial E[\widetilde{ROE}]/\partial d_1$, and $\partial V/\partial d_1$ for different d_1 values. The present (market) values of debt and equity, D and S, are calculated from (26) and (35). The present value of the project is $V = D + S$, and *NPV* $= V - A$. Of course, these are only valid for $S \geq 0$. The percentage of debt financing used in the project, D/A, is the ratio between the present value of the project's debt (*i.e.*, the amount of money debtholders will provide to the project) and the initial cost of the project.

The effective return on debt, $E[\tilde{r}_D]$, is the expected return for the debtholders. It can be determined by substituting (25) into (1). Thus, $D(1+E[\tilde{r}_D])$ is the repayment amount debtholders expect to receive at the end of the period. The promised return on debt, *Int*, is the interest rate debtholders would charge the owning company in order to lend them D. *Int* is calculated as the ratio between d_1 and D, minus one.

The required return on equity, $E[\tilde{r}_S]$, is the return investors would expect to receive if they had invested in an openly-traded asset that presents the same degree of risk as the privately-financed project (*i.e.*, $\beta_{\text{asset}} = \beta_{\text{project}}$). The expected return on equity investment, $E[\widetilde{Roe}]$, is the ratio between the expected end-of-period value of the project after all obligations have been paid and the amount infused by investors at the beginning of the period, minus one. The rates of change, $\partial D/\partial d_1$, $\partial E[\widetilde{ROE}]/\partial d_1$, and $\partial V/\partial d_1$, are calculated from (37), (46), and (47) respectively.

Figure 3 shows D, S, and V, as d_1 increases. According to (16), the value of the debt is the amount of money debtholders expect to receive at the end of the period minus the systematic risk premium on the project's debt (*i.e.*, the amount lenders charge to buy part of the project's systematic operating risk premium from the owning company), divided by R_f. As long as $d_1 < d_1^c$, any increment on the promised debt amount, increases the amount debtholders expect to receive at the end of the period more than it increases the amount they charge to take the risk from the owning company. Thus, any increment in d_1 would increase both the nominal interest rate *Int* and the loan amount D; that is, in the interval $(0, d_1^c)$, the systematic risk premium on the project's debt would never dominate the expected debt repayment amount.

At $d_1 = d_1^c$, the market value of the debtholders' holdings reaches a maximum, therefore D^c is the maximum amount of money the owning company can borrow from debtholders (*i.e.*, the debt capacity of the project). At this point, a small increase in d_1 is completely offset by an appropriate increase in *Int* leaving D^c constant. If $d_1 > d_1^c$, debtholders would decrease the

Table 2: Example Results (all $ in millions)

Promised debt amount d_1 (1)	Market value of debt D (2)	Market value of equity S (3)	Market value of project V (4)	Net Present Value *NPV* (5)	Debt financing D/A (6)	Effective return on debt $E[\tilde{r}_D]$ (7)	Promised return on debt *Int* (8)	Required return on equity $E[\tilde{r}_S]$ (9)	Return on equity investment $E[\widetilde{Roe}]$ (10)	$\partial D/\partial d_1$ (11)	$\partial E[\widetilde{ROE}]/\partial d_1$ (12)	$\partial V/\partial d_1$ (13)
0	0	2,293	2,293	123	0.000	0.060	0.060	0.111	0.174	0.94	5.79E-05	0.02
174	164	2,132	2,295	125	0.075	0.061	0.061	0.115	0.184	0.94	6.70E-05	0.02
347	327	1,971	2,298	128	0.151	0.061	0.062	0.119	0.197	0.94	7.79E-05	0.01
521	490	1,810	2,300	130	0.226	0.062	0.064	0.125	0.212	0.94	9.06E-05	0.01
694	651	1,649	2,301	131	0.300	0.062	0.066	0.131	0.229	0.93	1.05E-04	0.00
714	670	1,631	2,301	131	0.309	0.063	0.066	0.132	0.231	0.93	1.07E-04	0.00
868	811	1,489	2,300	130	0.374	0.064	0.070	0.139	0.248	0.91	1.21E-04	-0.01
1,042	968	1,329	2,297	127	0.446	0.066	0.076	0.149	0.271	0.89	1.36E-04	-0.02
1,215	1,119	1,172	2,291	121	0.516	0.069	0.086	0.161	0.295	0.85	1.47E-04	-0.04
1,389	1,264	1,018	2,282	112	0.582	0.073	0.099	0.176	0.321	0.80	1.47E-04	-0.07
1,562	1,397	871	2,268	98	0.644	0.078	0.118	0.194	0.345	0.73	1.24E-04	-0.10
1,736	1,517	732	2,249	79	0.699	0.085	0.145	0.215	0.362	0.64	5.65E-05	-0.13
1,800	1,556	684	2,240	70	0.717	0.088	0.157	0.224	0.364	0.60	1.49E-05	-0.14
1,910	1,619	605	2,223	53	0.746	0.093	0.180	0.241	0.360	0.53	-8.34E-05	-0.17
2,083	1,700	491	2,191	21	0.783	0.102	0.225	0.270	0.327	0.41	-3.19E-04	-0.21
2,179	1,736	434	2,170	0	0.800	0.107	0.255	0.289	0.289	0.34	-4.87E-04	-0.23
2,257	1,759	392	2,151	-19	0.811	0.112	0.283	0.305	0.245	0.28	-6.37E-04	-0.25
2,430	1,796	308	2,104	-66	0.828	0.122	0.353	0.343	0.106	0.15	-9.54E-04	-0.29
2,604	1,812	238	2,050	-120	0.835	0.132	0.437	0.386	-0.079	0.03	-1.14E-03	-0.33
2,659	1,812	219	2,031	-139	0.835	0.135	0.467	0.401	-0.143	0.00	-1.16E-03	-0.34
2,778	1,809	181	1,990	-180	0.834	0.142	0.536	0.435	-0.280	-0.06	-1.14E-03	-0.36
2,951	1,792	135	1,927	-243	0.826	0.150	0.647	0.489	-0.467	-0.13	-9.91E-04	-0.36
3,125	1,766	99	1,865	-305	0.814	0.155	0.769	0.552	-0.621	-0.17	-7.80E-04	-0.35
3,298	1,736	70	1,806	-364	0.800	0.159	0.900	0.624	-0.738	-0.18	-5.79E-04	-0.32
3,472	1,706	48	1,754	-416	0.786	0.160	1.035	0.708	-0.824	-0.17	-4.14E-04	-0.32

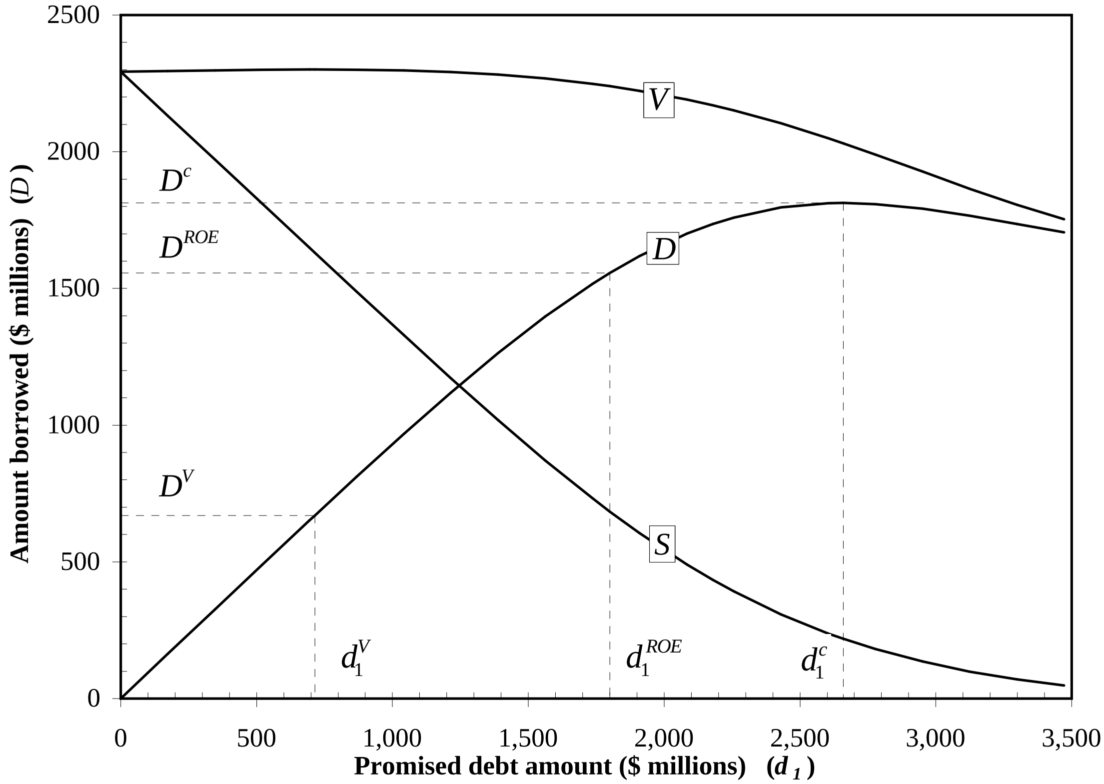

Figure 3: Present Values of V, D, and S as Functions of d_1

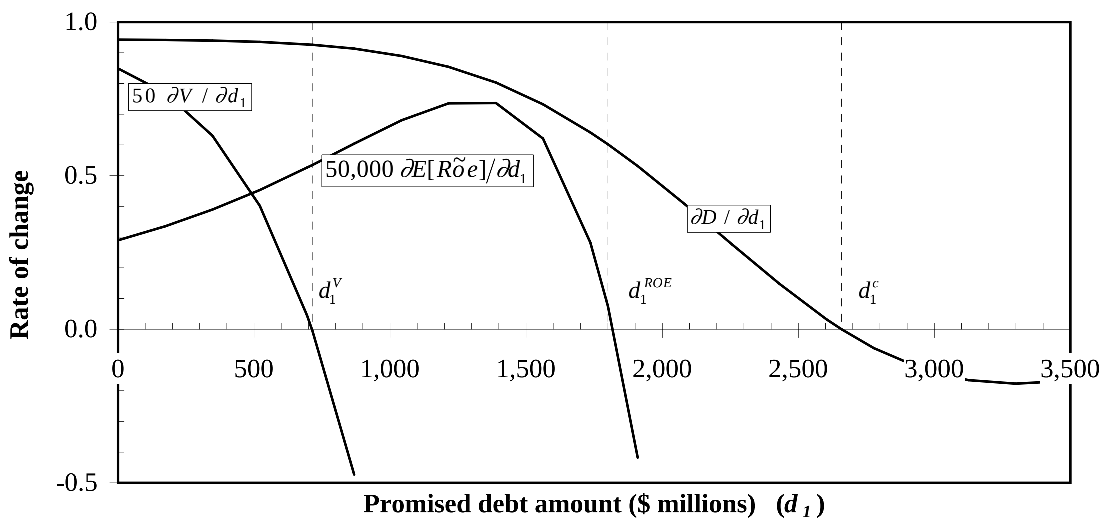

Figure 4: Rates of Change of D, $E[\widetilde{Roe}]$, and V as Functions of d_1

amount they would lend to the owning company because the amount charged by them to buy the risk from the company would always dominate the amount they expect to receive from debt repayment; that is, an increase in d_1 results in such a large increase in *Int* that the value of the debt D actually decreases.

As d_1 increases, the value of the equity, S, decreases because: (i) the probability of bankruptcy increases and (ii) the amount infused by investors decreases. As the probability of bankruptcy increases, the likelihood that a project generates enough income to distribute earnings to investors, after paying for all financial obligations, decreases. If $d_1 > d_1^c$, investors would, theoretically, infuse more equity to finance the project (since D now decreases). However, the probability of bankruptcy more than offsets this increase in equity infusion and S would still decrease as d_1 increases past d_1^c.

Figure 3 shows that the value of the owning company, V, first increases slightly as d_1 increases, reaches a maximum at d_1^V and then decreases. Figure 4 shows the point d_1^V where $\partial V/\partial d_1 = \partial NPV/\partial d_1 = 0$ and thus, illustrates the existence a debt financing amount, D^V, that maximizes the value of the owning company as well as the *NPV* of the project.

The dashed lines in Figures 5 and 6 correspond to the expected values of the returns on the debt and equity of an openly-traded asset that has the same risk class as the privately-financed project. The solid lines represent the investors' expected rate of return on equity, the interest rate charged by debtholders, and the project's *NPV*. Figure 5 shows that the difference between *Int* and $E[\tilde{r}_D]$ widens as d_1 increases, and thus, illustrates how the premium charged by lenders to take some of the net income risks from the investors increases as d_1 increases. It also shows that the promised debt amounts d_1^V and d_1^{ROE} do indeed maximize the project's NPV and the investors' expected return on equity. Figure 3 shows the associated optimal debt amounts D^V and D^{ROE} (depending on which objective one chooses to maximize).

Moreover, Figures 5 and 6 show that $E[\widetilde{Roe}]$ can be larger, equal or smaller than $E[\tilde{r}_S]$. At point ''*Z*,'' $E[\widetilde{Roe}] = E[\tilde{r}_S]$ and the project has $NPV = 0$. Promised debt amounts smaller than the one corresponding to ''*Z*'' yield $E[\widetilde{Roe}] > E[\tilde{r}_S]$; consequently, the project has a positive *NPV* and investors earn more than the return required by the market (as they should because they are the ones that create value by making the project a reality).

Figure 6 is similar to Figure 5 but uses ''percent debt financing'' (*i.e.*, D/A) as the x-axis

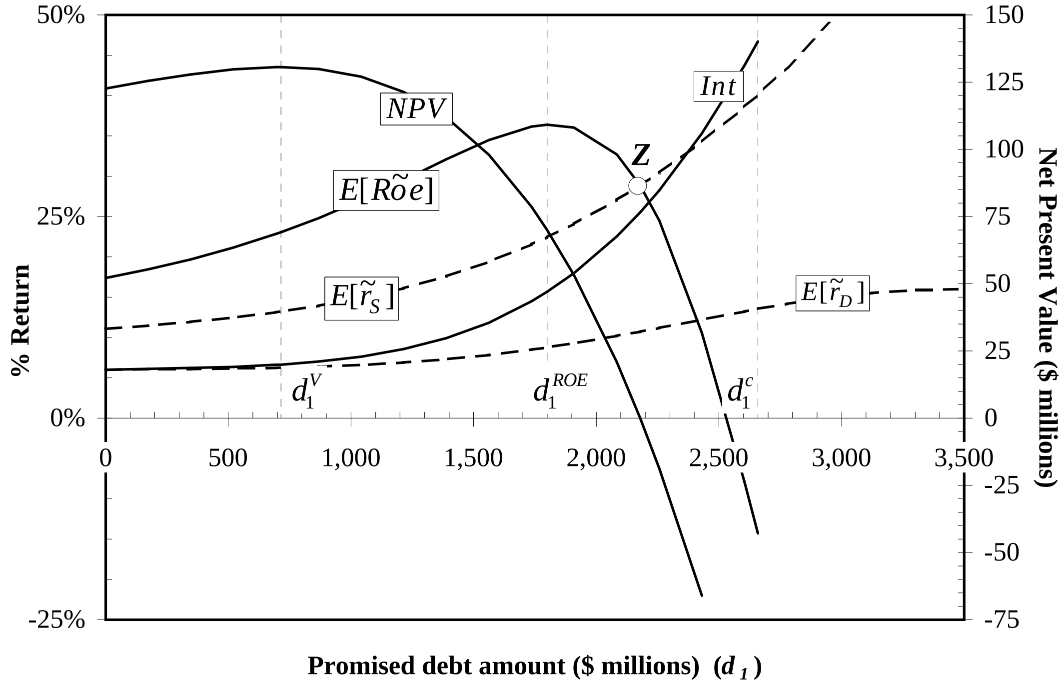

Figure 5: $E[\tilde{r}_D]$, *Int*, $E[\tilde{r}_S]$, $E[\widetilde{Roe}]$, and *NPV* as Functions of d_1

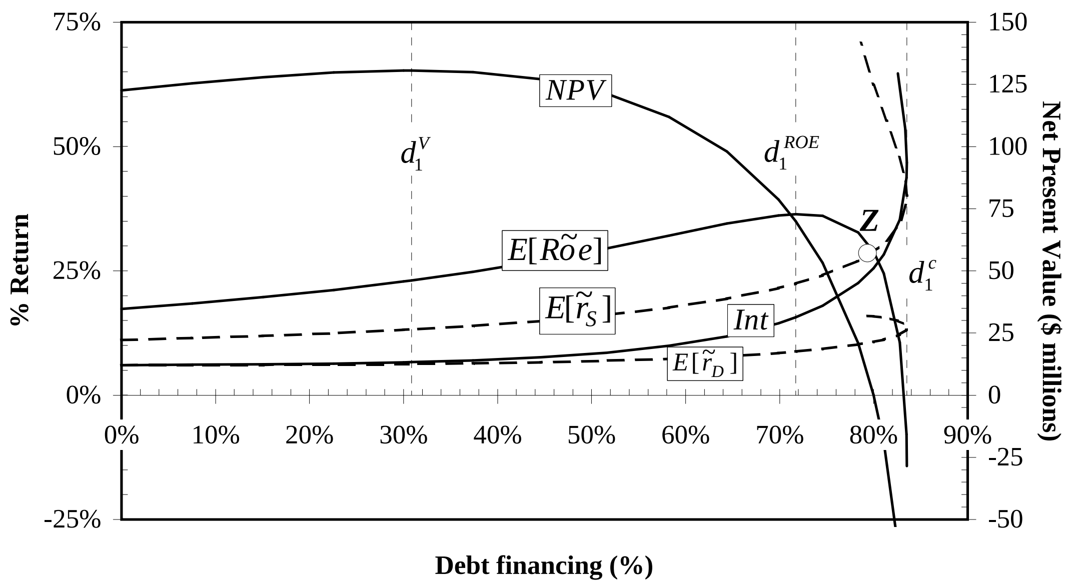

Figure 6: $E[\tilde{r}_D]$, *Int*, $E[\tilde{r}_S]$, $E[\widetilde{Roe}]$, and *NPV* as Functions of Percentage of Debt Financing

instead of the promised debt amount d_1. Thus, it illustrates the existence of a ‘‘***debt-capacity frontier***’’ at d_1^c, that limits the borrowing of the owning company. The reason, of course, is the existence of a maximum debt D^c as shown in Figure 3. For example, if 78.3% of the project is financed through debt, then lenders would provide $1,700 M and would require a promised debt amount of either $2,083 M (if $d_1 < d_1^c$) or $3,510 M (if $d_1 > d_1^c$). Thus, depending on whether d_1 is smaller or larger than d_1^c, lenders would charge an interest of either 27% or 106% and would have an expect return of either 10.2% or 16.0%. It is obvious that, from the owning company's viewpoint, it would always prefer to promise to pay less for the money it borrows. Therefore, the company will never borrow more than the debt capacity of the project as it would always put it in a worse situation than if it borrows up to capacity. Similarly, the lenders would never be able to expect a return on the debt greater than the $E[\tilde{r}_D]$ that occurs at d_1^c.

Figure 6 also shows that both the project's *NPV*and $E[\widetilde{Roe}]$ decline rapidly as D approaches D^c. Thus, both objectives can be quite sensitive to the amount of debt used to finance the project and the owning company should take great care to avoid excessive borrowing even if lenders are willing to provide the debt.

Conclusion

Private promotion of projects is an alternative arrangement for developing and implementing many types of projects that range from civil infrastructure works to industrial facilities. The involvement of the private sector can provide two major benefits: (a) gains in efficiency arising from the business expertise offered by the private sector (*e.g.*, innovation, marketing and management skills) and greater incentives for the control of construction, operating, and maintenance costs; and (b) the provision of additional finance for economically justifiable projects. A more detailed discussion of the many benefits of concession-financed projects appears in (Dias, 1994).

The creation of a single-project company allows off-balance-sheet financing, which is advantageous to corporate equityholders because of their limited liability. This situation creates risky and ‘‘expensive’’ debt because debtholders increase their risk premiums to account for the probability that the owning company defaults. We have shown that there is a limit to what owning companies can borrow to finance a project (debt capacity). If bankruptcy is costly, this limit is reached before bankruptcy becomes certain. We have also shown that if equityholders want to maximize the project's net present value, or the return on their investment, then they should borrow less than the available debt capacity. We have also compared the objective of maximizing equity returns with the more traditional objective of maximizing equityholders' wealth (*i.e.*, net present value). From this it was shown that the capital structure used to maximize returns would allow more debt financing than the more traditional objective of maximizing wealth and *NPV*. Most importantly, we have illustrated that the project's *NPV* and the equityholders return can be quite sensitive to the selected debt-equity ratio and decline rapidly as the owning company borrows more than the optimal amount in an attempt to reach the project's debt capacity level D^c. Thus, the issue of optimal capital structure merits significant attention at the project evaluation phase.

A one-period project was assumed in order to keep the mathematical analysis as simple as possible. The same general conclusions, however, are also valid for multi-period projects. The analysis, in this case, is best undertaken using numerical methods (a simple spreadsheet model may be sufficient) that apply the basic mathematical results presented above over many periods. An extension of these concepts to the evaluation of the effect of government guarantees on project financing appears in (Dias 1994).

Acknowledgments

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Appendix I. Basic Formulas for Normally Distributed Random Variables

Partial moments of Normally distributed random variables

The first partial moment of a Normal distribution is given by

$$E_a^b[\tilde{X}] = \int_a^b \tilde{X} f_X(\tilde{X}) d\tilde{X} \quad (54)$$

$$= E[\tilde{X}](F_X(b) - F_X(a)) + \sigma_X^2 (f_X(a) - f_X(b)) \quad (55)$$

Proof:

$$\int_a^b \tilde{X} f_X(\tilde{X}) d\tilde{X} = \int_a^b \frac{\tilde{X}}{\sqrt{2\pi}\sigma_X} e^{-\frac{1}{2}\left(\frac{\tilde{X}-m_X}{\sigma_X}\right)^2} d\tilde{X} \quad (56)$$

Substituting $\tilde{X}$ by $m_X + \tilde{U}\sigma_X$ and $d\tilde{X}$ by $\sigma_X d\tilde{U}$ in the above equation we have:

$$\begin{aligned}
\int_a^b \tilde{X} f_X(\tilde{X}) d\tilde{X} &= \int_{\frac{a-m_X}{\sigma_X}}^{\frac{b-m_X}{\sigma_X}} \frac{m_X + \tilde{U}\sigma_X}{\sqrt{2\pi}} e^{-\frac{1}{2}\tilde{U}^2} d\tilde{U} \\
&= \frac{m_X}{\sqrt{2\pi}} \int_{\frac{a-m_X}{\sigma_X}}^{\frac{b-m_X}{\sigma_X}} e^{-\frac{1}{2}\tilde{U}^2} d\tilde{U} + \frac{\sigma_X}{\sqrt{2\pi}} \int_{\frac{a-m_X}{\sigma_X}}^{\frac{b-m_X}{\sigma_X}} \tilde{U} e^{-\frac{1}{2}\tilde{U}^2} d\tilde{U} \\
&= m_X \left[F_U\left(\frac{b-m_X}{\sigma_X}\right) - F_U\left(\frac{a-m_X}{\sigma_X}\right)\right] + \\
&\quad + \sigma_X^2 \left[\frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{1}{2}\left(\frac{a-m_X}{\sigma_X}\right)^2} - \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{1}{2}\left(\frac{b-m_X}{\sigma_X}\right)^2} \right] \\
&= m_X(F_X(b) - F_X(a)) + \sigma_X^2 (f_X(a) - f_X(b)) \quad (57)
\end{aligned}$$

The second partial moment of a Normal distribution is given by

$$\begin{aligned}
E_a^b[\tilde{X}^2] &= \int_a^b \tilde{X}^2 f_X(\tilde{X}) d\tilde{X} \\
&= (E[\tilde{X}] + \sigma_X^2)(F_X(b) - F_X(a)) + \sigma_X^2 \{E[\tilde{X}](f_X(a) - f_X(b)) + a f_X(a) - b f_X(b)\} \quad (58)
\end{aligned}$$

The proof of equation (58) follows a reasoning similar to the one used to prove equation (55). Alternatively, Winkler *et.al.* (1972, pp. 294) provide an equation that can be used to calculate the partial n^{th} moment of a Normal distribution,

$$\begin{aligned} E^b_{-\infty}[\tilde{X}^n] &= \int_{-\infty}^{b} \tilde{X}^n f_X(\tilde{X}) d\tilde{X} \\ &= -\sigma_X^2 b^{n-1} f_N(b) + (n-1)\sigma_X^2 E^b_{-\infty}[\tilde{X}^{n-2}] + E[\tilde{X}]\, E^b_{-\infty}[\tilde{X}^{n-1}] \qquad (59) \end{aligned}$$

For $n = 2$:

$$\begin{aligned} E^b_{-\infty}[\tilde{X}^2] &= \int_{-\infty}^{b} \tilde{X}^2 f_X(\tilde{X})\, d\tilde{X} \\ &= -\sigma_X^2 b f_X(b) + \sigma_X^2 E^b_{-\infty}[\tilde{X}^0] + E[\tilde{X}]\, E^b_{-\infty}[\tilde{X}^1] \\ &= -\sigma_X^2 b f_X(b) + \sigma_X^2 F_X(b) + E[\tilde{X}] \int_{-\infty}^{b} \tilde{X} f_X(\tilde{X})\, d\tilde{X} \qquad (60) \end{aligned}$$

Therefore,

$$\begin{aligned} E^b_a[\tilde{X}^2] &= \sigma_X^2\,(a f_X(a) - b f_X(b)) + \sigma_X^2\,(F_X(b) - F_X(a)) + E[\tilde{X}] \int_{a}^{b} \tilde{X} f_X(\tilde{X})\, d\tilde{X} \qquad (61) \\ &= (E[\tilde{X}] + \sigma_X^2)(F_X(b) - F_X(a)) + \sigma_X^2 \{E[\tilde{X}](f_X(a) - f_X(b)) + a f_X(a) - b f_X(b)\} \qquad (62) \end{aligned}$$

Conditional expected values for jointly Normally distributed random variables

According to Benjamin and Cornell (1970, pp. 421) iff $\tilde{X}$ and $\tilde{R}_m$ are jointly Normally distributed, then

$$\begin{aligned} E[\tilde{R}_m|\tilde{X} = x] &= \int_{-\infty}^{\infty} \tilde{R}_m\, f_{R_m|X}(\tilde{R}_m|\tilde{X} = x)\, d\tilde{R}_m \\ &= E[\tilde{R}_m] + \rho_{R_m,X} \frac{\sigma_{R_m}}{\sigma_X} \left(\tilde{X} - E[\tilde{X}]\right) \\ &= E[\tilde{R}_m] + \frac{\text{Cov}(\tilde{X}, \tilde{R}_m)}{\sigma_X^2} \left(\tilde{X} - E[\tilde{X}]\right) \qquad (63) \end{aligned}$$

Appendix II. Derivation of Auxiliary Mathematical Relationships

Determination of expected values

Given that δ_b and δ_q are Bernoulli random variables with a certain probability of success and that $\tilde{X}$ and $\tilde{R}_m$ are Normally distributed random variables then,

$$E[\delta_b] = P[\text{ Success}] = F_X(d_1) \tag{64}$$

$$E[\delta_q] = 1 - F_X\left(\frac{b_f}{1-b_v}\right) = 1 - F_X(b') \tag{65}$$

$$E[\delta_b\delta_q] = F_X(d_1) - F_X\left(\frac{b_f}{1-b_v}\right) = F_X(d_1) - F_X(b') \tag{66}$$

$$E[\delta_b\delta_q\tilde{X}] = \int_{-\infty}^{\infty} \delta_b\delta_q\tilde{X} f_X(\tilde{X}) d\tilde{X} = \int_{b'}^{d_1} \tilde{X} f_X(\tilde{X}) d\tilde{X} \tag{67}$$

The above equation is similar to (54). Therefore,

$$E[\delta_b\delta_q\tilde{X}] = E[\tilde{X}]\left(F_X(d_1) - F_X(b')\right) + \sigma_X^2\left(f_X(b') - f_X(d_1)\right) \tag{68}$$

$$\begin{aligned} E[\delta_b\delta_q\tilde{R}_m] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta_b\delta_q\tilde{R}_m\, f_{X,R_m}(\tilde{X},\tilde{R}_m)\, d\tilde{R}_m\, d\tilde{X} \\ &= \int_{b'}^{d_1} \delta_b\delta_q\, f_X(\tilde{X}) \int_{-\infty}^{\infty} \tilde{R}_m\, f_{R_m|X}(\tilde{R}_m|\tilde{X}=x)\, d\tilde{R}_m\, d\tilde{X} \end{aligned} \tag{69}$$

Substituting equation (63) into equation (69) and from the definitions given above:

$$\begin{aligned} E[\delta_b\delta_q\tilde{R}_m] &= E[\tilde{R}_m]\, E[\delta_b\delta_q] + \frac{\text{Cov}(\tilde{X},\tilde{R}_m)}{\sigma_X^2}\Big\{E[\tilde{X}]\left(F_X(d_1) - F_X(b')\right) + \\ &\quad +\sigma_X^2\left(f_X(b') - f_X(d_1)\right) - E[\tilde{X}]\left(F_X(d_1) - F_X(b')\right)\Big\} \end{aligned} \tag{70}$$

$$\begin{aligned} E[\delta_b\delta_q\tilde{X}\tilde{R}_m] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta_b\delta_q\tilde{X}\tilde{R}_m f_{X,R_m}(\tilde{X},\tilde{R}_m)\, d\tilde{R}_m\, d\tilde{X} \\ &= \int_{b'}^{d_1} \delta_b\delta_q\tilde{X} f_X(\tilde{X}) \int_{-\infty}^{\infty} \tilde{R}_m f_{R_m|X}(\tilde{R}_m|\tilde{X}=x)\, d\tilde{R}_m\, d\tilde{X} \end{aligned} \tag{71}$$

Substituting equations (63), (58) and (55) into equation (71) gives:

$$\begin{aligned} E[\delta_b \delta_q \tilde{X} \tilde{R}_m] &= E[\tilde{R}_m] \left\{ E[\tilde{X}] \left(F_X(d_1) - F_X(b') \right) + \sigma_X^2 \left(f_X(b') - f_X(d_1) \right) \right\} + \\ &\quad + \frac{\text{Cov}(\tilde{X}, \tilde{R}_m)}{\sigma_X^2} \left\{ \sigma_X^2 \left(b' f_X(b') - d_1 f_X(d_1) \right) + \sigma_X^2 \left(F_X(d_1) - F_X(b') \right) \right\} \end{aligned} \quad (72)$$

Determination of covariances

$$\text{Cov}(\delta_b \delta_q, \tilde{R}_m) = E[\delta_b \delta_q \tilde{R}_m] - E[\delta_b \delta_q]\, E[\tilde{R}_m] \quad (73)$$

Substituting equations (66) and (70) into equation (73) gives:

$$\begin{aligned} \text{Cov}(\delta_b \delta_q, \tilde{R}_m) &= \frac{\text{Cov}(\tilde{X}, \tilde{R}_m)}{\sigma_X^2} \sigma_X^2 \left(f_X(b') - f_X(d_1) \right) + E[\tilde{R}_m]\, E[\delta_b \delta_q] - E[\tilde{R}_m]\, E[\delta_b \delta_q] \\ &= \left(f_X(b') - f_X(d_1) \right) \ \text{Cov}(\tilde{X}, \tilde{R}_m) \end{aligned} \quad (74)$$

$$\text{Cov}(\delta_b \delta_q \tilde{X}, \tilde{R}_m) = E[\delta_b \delta_q \tilde{X} \tilde{R}_m] - E[\delta_b \delta_q \tilde{X}]\, E[\tilde{R}_m] \quad (75)$$

Substituting equations (70) and (72) into equation (75) gives:

$$\begin{aligned} \text{Cov}(\delta_b \delta_q \tilde{X}, \tilde{R}_m) &= E[\tilde{R}_m] \left\{ E[\tilde{X}] \left(F_X(d_1) - F_X(b') \right) + \sigma_X^2 \left(f_X(b') - f_X(d_1) \right) \right\} + \\ &\quad + \frac{\text{Cov}(\tilde{X}, \tilde{R}_m)}{\sigma_X^2} \left\{ \sigma_X^2 \left(b' f_X(b') - d_1 f_X(d_1) \right) + \sigma_X^2 \left(F_X(d_1) - F_X(b') \right) \right\} - \\ &\quad - E[\tilde{R}_m] \left\{ E[\tilde{X}] \left(F_X(d_1) - F_X(b') \right) + \sigma_X^2 \left(f_X(b') - f_X(d_1) \right) \right\} \\ &= \left\{ F_X(d_1) - F_X(b') + \left(b' f_X(b') - d_1 f_X(d_1) \right) \right\} \ \text{Cov}(\tilde{X}, \tilde{R}_m) \end{aligned} \quad (76)$$

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Appendix IV. Notation

The following symbols are used in this paper. Random variables are indicated by a tilde (∼) over their names.

A = Total project cost.

$\tilde{B}$ = Total cost of bankruptcy.

b' = $b_f/(1-b_v)$, net operating income ($\tilde{X}$) level at which the lenders are repaid nothing.

b_f = Fixed cost of bankruptcy.

b_v = Variable cost of bankruptcy.

D = Amount borrowed by owning company (project debt) while promising to repay d_1 at the end of period 1.

$\tilde{D}_1$ = Uncertain market value of a project's debt at the end of period 1.

d_1 = $D(1+\mathit{Int})$, amount promised to be paid to lenders at the end of period 1, (*i.e.*, principal + interest).

d_1^c = Amount of d_1 that maximizes the debt amount for a project (*i.e.*, the project reaches its debt capacity, D^c).

d_1^{ROE} = Amount of d_1 that maximizes the investors' return on equity.

d_1^V = Amount of d_1 that maximizes the project's value, *NPV*.

$E[\tilde{U}]$ = Expected value of the random variable U.

$F_U(u)$ = Cumulative distribution function (CDF) of random variable U evaluated at u.

$f_U(u)$ = Probability density function (PDF) of random variable U evaluated at u.

Int = Nominal interest rate charged by the lenders which creates a repayment obligation d_1 (at the end of period 1) on a loan D.

$\tilde{R}$ = $1 + \tilde{r}$, one plus the rate of return on a single-period project.

R_f = $1 + r_f$, one plus the risk-free rate of interest.

$\tilde{R}_m$ = $1 + \tilde{r}_m$, one plus the rate of return on the market.

$\widetilde{ROE}$ = $1 + \widetilde{Roe}$, one plus the investors' return on equity.

$\tilde{r}_D$ = The actual rate of return on the debt D (depends on how much of the promised amount d_1 is actually paid to the lenders at the end of period 1).

$\tilde{r}_S$ = The actual rate of return on equity S.

S = Present market value of a project's equity when the market is in equilibrium.

$\tilde{S}_1$ = Uncertain market value of a project's equity at the end of period 1.

T = Corporate income tax rate.

V = Present (actual) market value of a project when the market is in equilibrium.

$\tilde{V}_1$ = Uncertain market value of a project at the end of period 1.

$\tilde{X}$ = Net Operating Income (NOI) generated by the project at the end of period 1.

β = Asset beta as defined by the capital asset pricing model (CAPM).

δ_i = Several binary (0,1) auxiliary variables defined when introduced in the text.

λ = Market price of a unit of risk. The expected excess return that must be given up by an asset to reduce a unit of its risk.

σ_U^2 = Variance of the random variable U.