# Bidding Models — Symmetry and State of Information

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**Abstract:** Several competitive bidding models since the late 1960's have been founded, criticized, and even rejected, based on incorrectly stated arguments concerning the apparent symmetry of the competitive positions of the prospective bidders. The proliferation of these arguments has led to confusion about the validity of the proposed models. This paper: (1) Illustrates the correct use of symmetry in competitive bidding as a function of available information and control; (2) explains the differences among nonsymmetric states of information that lead to apparent symmetry in the case of only two competitors; (3) presents and explains the assumptions that lead to Friedman's general bidding model; (4) proves the probabilistic validity of Friedman's model by using correctly stated arguments of symmetry; and (5) provides the foundations for understanding the limitations of other models.

## Introduction

Several competitive bidding models since the late 1960's have been founded and/or debated using arguments based on symmetry in the competitive bidding process. The authors of these models and their followers have also questioned the validity of other models, and in particular Friedman's, on the premise that it violates the axioms of probability theory. In particular, they claim that it does not produce results supported by common sense when applied to symmetrical situations, where each competitor should appear equally likely to win (Carr 1982, 1983; Dixie 1974; Friedman 1956; Gates 1967; Rosenshine 1972).

This paper examines the issue of symmetry in a bidding competition by determining the conditional probability that a particular contractor wins over $n$ opponents, given an available state of information and control. It also illustrates that the basic errors in arguments based on symmetry stem from their failure to realize that:

1. The level of information and control that an individual, or contractor possesses does matter and must be explicitly considered when making arguments based on symmetry.

2. A perceived *equivalence* among *two* contractors can be based on three distinct and very different states of information:

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- Complete lack of information about the bidders' behavior.
- Probabilistic or statistical evidence that the bid distributions of all competitors are identical.
- Knowledge that a particular contractor is bidding at the median of his competitors' perceived bid distribution.

Friedman's assumptions are also presented. His model is developed from a competitor's and an outsider's perspective and its probabilistic (as opposed to modeling) validity is proven. A proof of Friedman's formulas based on correctly stated arguments of symmetry is also illustrated. A similar proof of mathematical validity can be extended to any model, irrespective of how accurately the model manages to represent the real world, provided of course, that its modeling assumptions do not violate the axioms of the mathematical theory on which it is based.

The principal objective of this paper is to present the correct application of symmetry to the competitive bidding problem, and not to advocate the *modeling* validity of Friedman's model or its use in actual practice. Any bidding model is at best an approximation of the real competitive bidding environment, and no model can be applicable to every conceivable situation. However, it is imperative that the assumptions and limitations of a model be fully understood and carefully evaluated before it is either adopted or rejected. This is particularly important because almost all publications on the subject have been very obscure about the underlying assumptions and the limitations of the proposed models.

As a result, the more general objective of this paper is to highlight important issues that need to be considered explicitly by both the proponents of bidding models and their potential users. The final decision as to whether a model is acceptable and applicable to a certain situation is the responsibility of the user who must be proficient both in probability theory and the theory of classical and Bayesian statistical inference.

## Basic Definitions

The notation adopted in this paper is to indicate random variables (and the identity of contractors) using capitalized symbols; decision variables and the values of random variables are indicated using lower-case symbols.

Consider the case of a particular contractor $A_0$ bidding on a new project against $n$ competitors $A_1, A_2, \ldots, A_n$. Let $C_i$ be the project cost estimate and $B_i$ the bid price for competitor $A_i$, $i = 0, 1, \ldots, n$. When considering his bidding strategy, contractor $A_0$ *knows* the value of his cost estimate $c_0$ and has complete control over his bid price $b_0$ (which is his decision variable of primary interest). There is nothing uncertain about either of these quantities, even though the *actual* cost of the project, given that $A_0$ will be the low bidder, is of course uncertain.

When preparing his cost estimate, contractor $A_0$ has to make a multitude of decisions: select crew compositions, estimate quantities, estimate production etc. In fact, the whole estimating process is a series of decisions based on available information. A change in any of these decisions will immediately change the cost estimate $c_0$.

After establishing his cost estimate, contractor $A_0$ selects a markup $m_0$ which he then adds to his cost estimate $c_0$ to arrive at a bid price $b_0$ (i.e. $b_0 = c_0 + m_0$). Equivalently, he may select his *firm's bid-to-cost ratio* $\text{fbc}_0$ (i.e. $\text{fbc}_0 = 1 + m_0/c_0$) and multiply it by his cost estimate $c_0$:

$$b_0 = \text{fbc}_0 c_0 \tag{1}$$

Contractor $A_0$ has complete control over the values of $c_0$, $m_0$, $\text{fbc}_0$, and $b_0$, all of which are decision variables even if their values have not been determined yet. Since the objective of any bidding model is to help contractor $A_0$ select the value of his bid $b_0$, it is typically assumed that the value of the cost estimate $c_0$ is given. This observation is fundamental in understanding the assumptions of any bidding model.

$A_0$ does not know the competitors' cost estimates $C_i$ and bids $B_i$ and given his state of information considers these to be random variables. From $A_0$'s perspective, $C_i$ and $B_i$ are random variables even though $C_i$ and $B_i$ are decision variables to the corresponding contractor $A_i$. Furthermore, $A_0$'s state of information, and hence his perspective, do not change depending on whether contractor $A_i$ has, or has not, determined and thus fixed the values of $C_i$ and $B_i$ as of yet.

Let the ratio of contractor $A_i$'s bid $B_i$ divided by contractor $A_0$'s cost estimate $c_0$ be denoted by $X_i$:

$$X_i = \frac{B_i}{c_0} \quad i = 1, 2, \ldots, n \tag{2}$$

For contractor $A_0$ the ratio $x_0 = b_0/c_0$ is a decision variable equal to his firm's bid-to-cost ratio $\text{fbc}_0$. From $A_0$'s point of view, the $X_i$ ($i = 1, \ldots, n$) are random variables representing the *apparent* bid-to-cost ratios $\text{AFBC}_i$ for each of the $n$ other competitors. These are apparent FBC's because the denominator of $X_i$ is $c_0$ and not $C_i$ (the unknown cost estimate that $A_i$ uses).

Each of the $n$ opponents' bids $B_i$ is equal to:

$$B_i = \text{FBC}_i C_i \quad i = 1, 2, \ldots, n \tag{3}$$

and hence:

$$X_i = \text{FBC}_i \frac{C_i}{c_0} \quad i = 1, 2, \ldots, n \tag{4}$$

For a given $c_0$, the random variable $X_i$ is the product of the two random variables $\text{FBC}_i$ and $C_i$ divided by the constant $c_0$, and thus the variability of its distribution includes both the variability of competitor $A_i$'s markup (through his $\text{FBC}_i$) and the variability of his unknown (to $A_0$) cost estimate $C_i$.

## The General Probability of Winning

The most important random variable of interest to contractor $A_0$ is the minimum of his $n$ competitors' apparent bid-to-cost ratios, $X_{min}$:

$$X_{min} = \min\{X_1, X_2, \ldots, X_n\} \tag{5}$$

The complement of the cumulative distribution function of $X_{min}$ equals the probability that contractor $A_0$ will indeed be the low bidder:

$$\begin{aligned} P[A_0 \text{ wins } | x_0, c_0] &= P[X_{min} > x_0 | c_0, x_0] \\ &= 1 - F_{X_{min}|x_0,c_0}(x_0) \end{aligned} \tag{6}$$

It is important to notice that the probability that $A_0$ wins is obviously a function of the selected bid-to-cost ratio $x_0$. Furthermore, this probability also depends on the cost estimate $c_0$. Since each of the random variables $X_i$ is by definition a function of $A_0$'s cost estimate $c_0$, the minimum of the $X_i$, $X_{min}$, is also a function of $c_0$. As a result, the probability distribution of $X_{min}$ should, in general, depend on $c_0$.

The estimation of the conditional probability density function of $X_{min}$ is the most difficult objective of any bidding model. It is also the most controversial, because it obviously requires assumptions that are not only reasonable from a modeling point of view, but are also compatible with the data available for statistical estimation.

## Friedman's Assumptions

In order to determine the conditional distribution of $X_{min}$ from contractor $A_0$'s perspective, Friedman has proposed the following assumptions:

1. The probability distribution of each apparent bid-to-cost ratio $X_i$ $(i = 1, 2, \ldots, n)$ does not depend on the values of $x_0$ and $c_0$ that $A_0$ chooses:

$$P[X_i < x_i | x_0, c_0] = P[X_i < x_i] \quad i = 1, 2, \ldots, n \tag{7}$$

2. The random variables $X_i$ $(i = 1, 2, \ldots, n)$ representing the apparent bid-to-cost ratios for the $n$ other competitors are mutually independent:

$$P[\bigcap_{i=1}^{n}(X_i < x_i)|c_0] = \prod_{i=1}^{n} P[X_i < x_i | c_0] \tag{8}$$

In order for these assumptions to be meaningful, it is also necessary to impose the implicit and often ignored condition that $A_0$ has determined an estimated cost $c_0$ (irrespective of its specific value). Otherwise, the ratios $X_i$ do not exist.

The rationale behind these assumptions is that since all $n + 1$ competitors, including $A_0$, are bidding on the same project, it is evident that (from $A_0$'s perspective) the probability distributions of the $n$ opponents' bid prices $B_i$ $(i = 1, 2, \ldots, n)$ do depend on the value of $c_0$ and on the apparent bid-to-cost ratios (i.e. $c_0$ is an estimate of the dollar magnitude of the project's cost and hence of the range of the expected bid prices). By the same token, the probability distribution of any bid $B_j$ is not independent of the value of some other bid $B_i$. If we imagine that $A_0$ learns the values of his competitors' bids one at a time, then the distribution of the bid to be announced next is a function of the bid prices already announced.

In order to eliminate these dependencies, Friedman proposes that we standardize each of the opponents' bids $B_i$ by taking its ratio with respect to contractor $A_0$'s cost estimate $c_0$. The resulting ratios have a probability distribution which is assumed independent of $c_0$. Furthermore, these ratios are also assumed to be mutually independent.

Both of these assumptions are necessary in order to develop a general bidding model that does not depend on the specific value of $c_0$ (which varies from contractor to contractor and from project to project) and for which *uncorrelated* data can be obtained from past experience. Other standardizations are also possible. For example, when analyzing the competitive bidding process from an owner's perspective, the author proposed that each bid $B_i$ be standardized by taking its ratio to the engineer's cost estimate (since the contractors' cost estimates were not available) (Ioannou, 1984). This assumption leads to a model identical to Friedman's.

## The Corollaries of Friedman's Assumptions

The first of Friedman's assumptions also implies two very important corollaries (mathematical consequences):

1. Unless contractor $A_0$ has already determined and thus fixed his cost estimate $c_0$, the probability distribution of any competitor's bid $B_i$ does not exist. In other words, if $A_0$ does not know what his cost estimate $c_0$ is, then the probability density function of the bid price $B_i$ equals zero: $f_{B_i}(b_i) = 0$, for all values $b_i$.

2. Similarly, if contractor $A_0$ knows the exact value of the bid-to-cost ratio $\text{FBC}_i$ that competitor $A_i$ uses, but has not already determined and thus fixed his cost estimate $c_0$, then the probability distribution of competitor $A_i$'s cost estimate $C_i$ does not exist: $f_{C_i}(c_i) = 0$, for all values $c_i$.

Both of these corollaries mean that the only absolute predictor available to $A_0$ for determining the range of the distribution of any bid $B_i$ is his own cost estimate $c_0$. If $A_0$ already knows $\text{FBC}_i$, then the same, of course, is true for the distribution of any cost estimate $C_i$. In the absence of this predictor, $A_0$ is completely ignorant about the characteristics of the project that contractor $A_i$ is bidding on, and thus any value $b_i$ is equally likely. Since $B_i$ can assume an infinity of values $b_i$, its probability density function must be zero for all $b_i$. This state of information is exactly what Bayesian statisticians call a *vague prior*.

The significance of these corollaries stems from the fact that unless the value of $c_0$ is considered fixed (but not necessarily known), Friedman's assumptions are not mathematically valid. However, this is not a limitation if these assumptions are applied *after* the cost estimate $c_0$ has been established. Under this condition, Friedman considers that (from a statistical point of view) the decision variable $c_0$ is a *sufficient* predictor of a project's cost. In other words, $c_0$ incorporates *all* the information available to contractor $A_0$ concerning *his own* cost on a particular future project.

## Friedman's Bidding Model

Friedman's bidding model gives the probability that $A_0$ wins, given that he has chosen a particular bid-to-cost ratio $x_0$. Starting from equation (6) and applying the above assumptions:

$$\begin{aligned} P[A_0 \text{ wins } | x_0, c_0] &= P[X_1 > x_0 \bigcap X_2 > x_0 \bigcap \ldots \bigcap X_n > x_0 | x_0, c_0] \\ &= P[\bigcap_{i=1}^{n} (X_i > x_0) | x_0, c_0] \\ &= \prod_{i=1}^{n} P[X_i > x_0 | x_0, c_0] \end{aligned}$$

$$= \prod_{i=1}^{n}[1 - F_{X_i|x_0,c_0}(x_0)]$$
$$= \prod_{i=1}^{n}[1 - F_{X_i}(x_0)] \tag{9}$$

The probability that $A_0$ wins is also shown to be dependent on $c_0$ in order to emphasize that the probability of submitting the low bid requires the existence of a bid distribution for a *particular* project, which in turn requires the existence of $c_0$ for that particular project. It is important to notice that expression (9) is applicable only when $x_0$ and $c_0$ are either known or controlled, and as a result, it can only be used by one of the competitors. It cannot be used by an outsider, unless the outsider happens to know the decisions of a particular contractor.

## Friedman's Model — The Case of Average Competitors

Friedman's model is simplified even further if contractor $A_0$ considers all his competitors to be *equivalent*. This assumption may be based on:

1. Lack of information about the bidding behavior of each of the opponents, in which case all are considered equivalent simply because $A_0$ does not have the data to discriminate between them.

2. Sufficient information about the bidding behavior of each opponent not only exists, but also suggests that the apparent bid-to-cost ratios for all $n$ opponents have the same probability distribution.

In either of these cases, the random variables $X_i$ ($i = 1, 2, \ldots, n$) can be considered independent and identically distributed (*iid*). Under this assumption, the probability density functions $f_{X_i}(x_i)$ and cumulative distribution functions $F_{X_i}(x_i)$ are given by the same generic functions $f_X(x)$ and $F_X(x)$. Thus, the probability that $A_0$ wins becomes:

$$P[A_0 \text{ wins } | x_0, c_0] = \prod_{i=1}^{n}[1 - F_{X_i}(x_0)]$$
$$= [1 - F_X(x_0)]^n \tag{10}$$

## The Fallacies of Apparent Symmetry

The *equivalence* of contractors is a notion that appears often in the discussion of bidding models, especially when proposing arguments based on symmetry. As illustrated below, its interpretation varies depending on the assumed state information, a fact that has not always received the proper attention in the literature.

False arguments based on symmetry typically make the following two assumptions: (a) contractor $A_0$ is bidding against $n$ equivalent competitors, and (b) he decides to bid at the median of his competitors' apparent bid-to-cost ratio distribution. Under these conditions, his probability of winning, according to equation (10), is:

$$\begin{aligned} P[A_0 \text{ wins } | x_0 = \text{ median of } X] &= [1 - F_X(\text{median of } X)]^n \\ &= [1 - 0.5]^n \\ &= 0.5^n \end{aligned} \tag{11}$$

This probability is obviously non-symmetric. For example, if the number of competitors $n$ opposing $A_0$ is 4, then the probability that contractor $A_0$ wins, according to Friedman's model, is $0.5^4 = 0.0625$. This is very different from $1/5 = 0.2$ which would be the result obtained from shortsighted symmetry, since 5 apparently equivalent competitors should have equal probabilities of winning.

The false symmetry-based argument can, in general, be presented as follows:

> If contractor $A_0$ bids at the median of the apparent bid-to-cost ratio distribution then he should have a 50% chance of winning against any single competitor $A_i$. Furthermore, since the bid-to-cost ratio distributions for the other $n$ competitors are identical, it is also equally likely that any contractor $A_i$ will win over any other single contractor $A_j$ $(i, j = 1, 2, \ldots, n)$. Since all $n + 1$ contractors are equally likely to win when faced with any other single opponent then they should all have the same probability of being the low bidder: $1/(n + 1)$.

A variation of this argument has also been suggested as the basis for a competitive bidding model:

> "Consider the situation where you are bidding against $n$ equally matched competitors. That is, at a certain profit markup you can expect to win over each of the other bidders exactly one half of the time. Then the chances of your winning over all $n$ competitors when bidding for the same contract is $1/(n + 1)$." (Gates 1967)

These arguments are in error because they ignore the non-symmetric state of information on which each probability of winning is based.

The probability that contractor $A_0$ will win over any other single opponent $A_i$ is based on perfect information about the fact that $A_0$ is bidding at the median of the bid-to-cost ratio distribution. Hence, the probability:

$$P[X_i > x_0 | x_0 = \text{ median of } X_i] = P[X_i > \text{ median of } X_i] = 0.5 \tag{12}$$

becomes either 0 or 1 if the value of $X_i$ becomes known to be either below or above the median of its distribution. Notice that the exact value of $X_i$ is not required.

In contrast, the probability that contractor $A_j$ will win over contractor $A_i$ is based on *equal lack* of information about the values of the *two* random variables $X_j$ and $X_i$ (or equivalently $B_j$ and $B_i$). Hence the probability:

$$P[X_i > X_j] = 0.5 \tag{13}$$

does not become either 0 or 1 if the value of $X_i$ becomes known to be either below or above the median of its distribution. The same is true for $X_j$. For example, if $X_i$ is known to be set at the 75th percentile, then:

$$P[X_i > x_0 | x_0 = \text{ median of } X_i, X_i = \text{ 75th percentile})] = 1 \tag{14}$$

$$P[X_i > X_j | X_i = \text{ 75th percentile}] = 0.75 \tag{15}$$

These probabilities are obviously non-symmetric because the original state of information on which each of them is based is fundamentally different.

The above fallacy based on apparent symmetry is very similar to the one concerning pairwise and mutual independence of random variables: both appeal to common sense. Students of probability theory typically jump to the conclusion that if a set of events $Z_i$ ($i = 1, 2, \ldots, n$) are pairwise independent:

$$P[Z_i \bigcap Z_j] = P[Z_i]P[Z_j] \quad i, j = 1, 2, \ldots, n; \quad i \neq j \tag{16}$$

then they should also be mutually independent:

$$P[Z_1 \bigcap Z_2 \bigcap \ldots \bigcap Z_n] = P[Z_1]P[Z_2] \ldots P[Z_n] \tag{17}$$

The reasons why this argument is incorrect are explained elsewhere (Benjamin 1970).

## Symmetry of Information — The Outsider's Perspective

The above symmetry-based arguments become valid only if special attention is paid to the symmetry of information.

Consider the perspective of an outsider who knows that $n+1$ equivalent contractors are bidding on a new project. This outsider does not know the estimated costs $C_i$, the bid-to-cost ratios $\text{FBC}_i$, or the bids $B_i$ of any of the $n+1$ competitors.

In this case, each of the $C_i$ and $X_i$ (including $C_0$ and $X_0$) are random variables, and as a result, it is impossible to apply Friedman's assumptions (and thus expressions (9) or (10)) directly. To solve this problem, we must assume that the outsider knows the cost estimate $c_0$ and the bid-to-cost ratio $x_0$ that contractor $A_0$ uses. The conditional probability that $A_0$ wins given this particular choice of $x_0$ and $c_0$ is given by expressions (9) and (10). To compute the marginal (unconditional) probability that $A_0$ (or any other contractor $A_i$) wins, we must use the total probability theorem to integrate the product of this conditional probability times the marginal probability distributions of $X_0$ and $C_0$, over all the values of $x_0$ and $c_0$:

$$\begin{aligned}
P[A_0 \text{ wins }] &= \sum_{\text{all } c_0} P[A_0 \text{ wins } | C_0 = c_0] P[C_0 = c_0] \\
&= \sum_{\text{all } c_0} \{ \sum_{\text{all } x_0} P[A_0 \text{ wins } | X_0 = x_0, C_0 = c_0] P[X_0 = x_0 | C_0 = c_0] \} P[C_0 = c_0] \\
&= \int_{c_0=-\infty}^{+\infty} \{ \int_{x_0=-\infty}^{+\infty} [1 - F_{X_i | X_0 = x_0, C_0 = c_0}(x_0)]^n f_{X_0 | C_0 = c_0}(x_0) dx_0 \} f_{C_0}(c_0) dc_0 \\
&= \int_{c_0=-\infty}^{+\infty} \{ \int_{x_0=-\infty}^{+\infty} [1 - F_{X_i}(x_0)]^n f_{X_0}(x_0) dx_0 \} f_{C_0}(c_0) dc_0 \\
&= \int_{c_0=-\infty}^{+\infty} \{ \int_{x_0=-\infty}^{+\infty} [1 - F_X(x_0)]^n f_X(x_0) dx_0 \} f_{C_0}(c_0) dc_0 \\
&= \int_{c_0=-\infty}^{+\infty} \{ [-\frac{[1 - F_X(x_0)]^{n+1}}{n+1}]_{x_0=-\infty}^{x_0=+\infty} \} f_{C_0}(c_0) dc_0
\end{aligned}$$

$$= \int_{c_0=-\infty}^{+\infty} (\frac{1}{n+1}) f_{C_0}(c_0) dc_0$$
$$= \frac{1}{n+1} \int_{c_0=-\infty}^{+\infty} f_{C_0}(c_0) dc_0$$
$$= \frac{1}{n+1} \qquad (18)$$

This result simply states that from the point of view of an outsider, the probability that $A_0$ (or any other contractor $A_i$) wins is:

$$P[A_i \text{ wins }] = \frac{1}{\text{Total number of competing contractors}} \qquad (19)$$

The same expression can be obtained by observing the symmetry of the outsider's state of information and thus arguing that under these conditions all contractors should appear (to the outsider) to be equally likely to win.

In the above application of the total probability theorem, the assumption that all contractors are *equivalent* is interpreted to mean that the $X_i$ (including $X_0$) are identically distributed and thus have the same cumulative distribution function $F_X(x)$ and probability density function $f_X(x)$. These distributions, as well as $f_{C_0}(c_0)$, are supposed to have been provided by the outsider and reflect his belief and state of information.

However, the derivation of equation (18) is independent of the type, or shape of the particular probability density functions involved in the proof. As a result, Friedman's assumptions (when employed from an outsider's perspective) yield a mathematical identity which is valid for all distributions, provided, of course, that the symmetry in the state of information is preserved.

Most importantly, however, it should be noted that Friedman's assumptions lead to different expressions for the probability that $A_0$ wins, depending on the assumed state of information and control, as illustrated by the differences in equations (9),(10) and (18).

## The Case of Equal Bid-to-cost Ratios

The symmetry of information and thus equation (19) should also apply to the case where an outsider knows that there are $n+1$ *equivalent* contractors bidding on a new project, and that all of them use the *same* bid-to-cost ratio $\text{FBC}_i = \text{FBC} = x_0$. For the brevity of notation let $G$ be the event:

$$G = \{\text{FBC}_i = \text{FBC} = x_0; \quad i = 0, 1, 2, \ldots, n\} \qquad (20)$$

In order to preserve the symmetry of information, it is also necessary to assume that the outsider does not know any of the contractors' estimates $C_i$, and thus any of their bids $B_i$. The only other assumption that does not violate symmetry is that all contractors have the same cost estimate $C_i$, and thus submit equal bids $B_i$, which would then result in a draw.

Obviously, the case where an outsider knows that all competitors use the same true bid-to-cost ratio (FBC) has little if any practical significance since it never occurs in practice. The only possible exception is the case of an owner who expects all FBC's to be very small and thus practically equal. However, the combination of this assumption with Friedman's assumptions presents a difficult mathematical problem that has some interesting theoretical subtleties. These difficulties have unfortunately led researchers to incorrect conclusions about the mathematical validity of his model (Carr 1982; 1983; Dixie 1974). The purpose of this section is to illustrate how Friedman's assumptions can be applied to this problem, and to show that the results are identical to those obtained by simple symmetry.

From contractor $A_0$'s perspective the apparent bid-to-cost ratios $X_i$ $(i = 1, 2, \ldots, n)$ are given by equation (4). In this case, the $X_i$ equal:

$$X_i = \frac{B_i}{c_0} = \frac{\text{FBC}_i}{c_0} C_i = \frac{\text{FBC}}{c_0} C_i = \frac{x_0}{c_0} C_i \quad i = 1, 2, \ldots, n \tag{21}$$

which is the product of the random variable $C_i$ times the ratio of the two known constants $x_0$ and $c_0$. The underlying variability of the distribution of $X_i$ has changed, since now it is solely due to the variability of $C_i$.

Given that each $\text{FBC}_i$ is equal to the same constant $x_0$, Friedman's first assumption implies that the probability distribution of the ratio $C_i/c_0$ $(i \neq 0)$ is *functionally* independent of the value $c_0$ that $A_0$ chooses:

$$\begin{aligned}
P[X_i > x_i | c_0] &= P[X_i > x_i] \\
P[x_0 \frac{C_i}{c_0} > x_i | c_0] &= P[x_0 \frac{C_i}{c_0} > x_i] \\
P[\frac{C_i}{c_0} > \frac{x_i}{x_0} | c_0] &= P[\frac{C_i}{c_0} > \frac{x_i}{x_0}] \\
P[\frac{C_i}{c_0} > r | c_0] &= P[\frac{C_i}{c_0} > r] \\
F_{\frac{C_i}{c_0}|c_0}(r) &= F_{\frac{C_i}{c_0}}(r)
\end{aligned} \tag{22}$$

Under the same conditions, Friedman's second assumption implies that for a given $c_0$ the ratios $C_i/c_0$ $(i \neq 0)$ are mutually independent:

$$\begin{aligned}
P[\bigcap_{i=1}^{n} (X_i > x_i) | c_0] &= \prod_{i=1}^{n} P[X_i > x_i | c_0] \\
P[\bigcap_{i=1}^{n} (x_0 \frac{C_i}{c_0} > x_i) | c_0] &= \prod_{i=1}^{n} P[x_0 \frac{C_i}{c_0} > x_i | c_0] \\
P[\bigcap_{i=1}^{n} (\frac{C_i}{c_0} > \frac{x_i}{x_0}) | c_0] &= \prod_{i=1}^{n} P[\frac{C_i}{c_0} > \frac{x_i}{x_0} | c_0] \\
P[\bigcap_{i=1}^{n} (\frac{C_i}{c_0} > r_i) | c_0] &= \prod_{i=1}^{n} P[\frac{C_i}{c_0} > r_i | c_0]
\end{aligned} \tag{23}$$

For contractor $A_0$, the probability distribution $F_{C_i/c_0}(r)$ of the ratio $C_i/c_0$ is only a function of $r$ (and not a function of $c_0$). As a result, $A_0$ can estimate the distribution of the ratio $C_i/c_0$, decide on his own cost estimate $c_0$ for a particular project, and use these to *derive* the distribution $F_{C_i}(c_i)$ for any other competitor's cost estimate $C_i$:

$$\begin{aligned}
P[C_i > c_i | c_0] &= P[\frac{C_i}{c_0} > \frac{c_i}{c_0} | c_0] \\
&= P[\frac{C_i}{c_0} > \frac{c_i}{c_0}] \\
&= 1 - F_{\frac{C_i}{c_0}}(\frac{c_i}{c_0})
\end{aligned} \tag{24}$$

And as a result:

$$F_{C_i|c_0}(c_i) = F_{\frac{C_i}{c_0}|c_0}(\frac{c_i}{c_0}) = F_{\frac{C_i}{c_0}}(\frac{c_i}{c_0}) \tag{25}$$

Notice that the chosen constant $c_0$ is a *scale parameter* of the *derived* distribution $F_{C_i|c_0}(c_i)$ and that the random variables $C_i/c_0$ and $C_i$ are *functionally* dependent. It is important here not to confuse functional dependence with stochastic or probabilistic dependence. Functional dependence can be thought of as perfect stochastic dependence: if the value of the independent random variable $C_i/c_0$ is known to be $r$, then the conditional distribution of $C_i$ becomes a unit mass at the value $c_0 r$ (Benjamin 1970).

From an outsider's perspective, however, $C_0$ is a random variable like the rest of the $C_i$. Given a *particular* project, the outsider can, in principle, assign a probability density function $f_{C_i}(c_i)$ describing the distribution of the estimated cost of any contractor $A_i$. Furthermore, since the outsider considers all competitors to be equivalent, the preservation of symmetry requires that all $C_i$ (including $C_0$) be identically distributed, following the same distribution $f_C(c_i)$. Given the distribution of every $C_i$, it follows that the conditional probability distribution of the ratio $C_i/c_0$ ($i \neq 0$) given $c_0$ can be mathematically derived from expressions (22) using $F_{C_i}(c_i)$ and the given value $c_0$:

$$\begin{aligned} P[\frac{C_i}{c_0} > r|C_0 = c_0] &= P[\frac{C_i}{c_0} > r] \\ &= P[C_i > c_0 r] \\ &= 1 - F_{C_i}(c_0 r) \end{aligned} \tag{26}$$

Thus, the outsider can use Friedman's assumption of independence to *compute* the *conditional* probability distribution of the ratio $C_i/c_0$ as a *derived* distribution based on $F_{C_i}(c_i)$ and the given scale parameter $c_0$:

$$F_{\frac{C_i}{c_0}|C_0=c_0}(r) = F_{\frac{C_i}{c_0}}(r) = F_{C_i}(c_0 r) \tag{27}$$

Notice that in this case the functional dependence between $C_i$ and $C_i/c_0$ is reversed, as reflected in the argument of $F_{C_i}(c_0 r)$. The independent random variable is $C_i$, whereas $C_i/c_0$ is the dependent variable. Furthermore, given that the outsider has already determined the *numeric* values of the parameters of the distribution $F_{C_i}(c_i)$ for a *given* project (e.g. for a Normal distribution he has determined the numeric values of the mean and variance), it follows that the distribution of $C_i/c_0$ does not have all its parameters fixed, since it also depends functionally on $c_0$. Thus, contractor $A_0$ (who considers $C_i$ to be a function of $C_i/c_0$) and the outsider have opposite (but equivalent) perspectives on the functional dependence between $C_i$ and $C_i/c_0$.

From an outsider's perspective, the probability that $A_0$ (or any other contractor $A_i$) wins, given the event $G$, is based on the application of the total probability theorem and the above observations.

Starting from first principles:

$$
\begin{aligned}
P[A_0 \text{ wins } |G] &= P[\bigcap_{i=1}^{n}(B_i > B_0)|G] \\
&= P[\bigcap_{i=1}^{n}(x_0 C_i > x_0 C_0)|G] \\
&= P[\bigcap_{i=1}^{n}(\frac{C_i}{C_0} > 1)] \\
&= \sum_{\text{all } c_0}\{P[\bigcap_{i=1}^{n}(\frac{C_i}{c_0} > 1)|C_0 = c_0]P[C_0 = c_0]\} \\
&= \sum_{\text{all } c_0}\{\prod_{i=1}^{n} P[\frac{C_i}{c_0} > 1|C_0 = c_0]P[C_0 = c_0]\} \\
&= \sum_{\text{all } c_0}\{\prod_{i=1}^{n} P[\frac{C_i}{c_0} > 1]P[C_0 = c_0]\} \\
&= \sum_{\text{all } c_0}\{\prod_{i=1}^{n} P[C_i > c_0]P[C_0 = c_0]\} \\
&= \int_{c_0=-\infty}^{+\infty}[1 - F_{C_i}(c_0)]^n f_{C_0}(c_0)dc_0 \\
&= \int_{c_0=-\infty}^{+\infty}[1 - F_C(c_0)]^n f_C(c_0)dc_0 \\
&= [-\frac{[1 - F_C(c_0)]^{n+1}}{n+1}]_{-\infty}^{+\infty} \\
&= \frac{1}{n+1} \qquad (28)
\end{aligned}
$$

This result is again identical to the one obtained by observing the symmetry of information and thus arguing that under these conditions all contractors should be equally likely to win.

In fact, the combination of (a) Friedman's assumptions, (b) the equality of FBC's, and (c) the equivalence of contractors (meaning that the $C_i$, including $C_0$, are identically distributed having the same probability density function $f_C(c)$) results in the following game (as viewed by an outsider): Each contractor $A_i$, out of $n + 1$ competitors, makes a single independent draw $C_i$ from the same distribution $f_C(c)$. The contractor who draws the smallest $C_i$ wins the project. Obviously, the *a priori* likelihood of each competitor winning is $1/(n + 1)$ based on simple symmetry. Notice that the function $f_C(c)$ represents the *outsider's* assessment of the expected distribution of estimated costs for the project. However, the final result is again a mathematical identity that does not depend on the specific form of this distribution. Furthermore, the symmetry-based argument does not require and is not contradicted by Friedman's assumptions of independence.

From the above results it is evident that as long as the outsider's assumed state of information is truly symmetric, Friedman's assumptions will always lead to the same conclusion: all competitors are equally likely to win, and the probability of each winning is $1/(n+1)$. Examples of other such cases are: all contractors have the same cost estimate equal to $c_0$, but unknown FBC$_i$'s; or, all contractors have the same FBC$_i$ whose exact value is unknown, etc.

## Symmetry of Information — A Contractor's Perspective

From contractor $A_0$'s perspective, the probability that some other competitor $A_1$ will be the low bidder (out of the $n$ equivalent competitors and where the index 1 is generic and is simply chosen for the ease of notation) is given by the total probability theorem, as follows:

$$\begin{aligned} P[A_1 \text{ wins}|x_0, c_0] &= P[x_0 > X_1 \bigcap X_2 > X_1 \bigcap \ldots \bigcap X_n > X_1 | x_0] \\ &= P[(x_0 > X_1) \bigcap (\bigcap_{j=2}^{n} X_j > X_1) | x_0] \\ &= \sum_{\text{all } x_1} \{P[(x_0 > x_1) \bigcap (\bigcap_{j=2}^{n} X_j > x_1) | x_0, X_1 = x_1] P[X_1 = x_1]\} \\ &= \sum_{\text{all } x_1} \{P[x_0 > x_1] \prod_{j=2}^{n} P[X_j > x_1] P[X_1 = x_1]\} \\ &= \int_{x_1=-\infty}^{+\infty} P[x_0 > x_1][1 - F_X(x_1)]^{n-1} f_X(x_1) dx_1 \\ &= \int_{x_1=-\infty}^{x_0} [1 - F_X(x_1)]^{n-1} f_X(x_1) dx_1 \\ &= [-\frac{[1 - F_X(x_1)]^n}{n}]_{-\infty}^{x_0} \\ &= \frac{1}{n}\{1 - [1 - F_X(x_0)]^n\} \end{aligned} \tag{29}$$

Notice that in the above expression the probability:

$$P[x_0 > X_1 | x_0, X_1 = x_1] = P[x_0 > x_1] = \begin{cases} 1 & \text{for } x_0 > x_1 \\ 0 & \text{for } x_0 < x_1 \end{cases} \tag{30}$$

equals either 1 or 0 depending on the given values of $x_0$ and $x_1$. This observation is reflected in the limits of integration.

From an outsider's perspective, the probability that $A_1$ wins, can be computed by applying the total probability theorem and using expression (29) instead of (10). The result is $1/(n+1)$, the same as given by expression (18).

## The Correct Application of Symmetry

Expression (29) can also be obtained more easily based on correctly stated arguments of symmetry. The probability that $A_0$ wins, given that he chooses a bid-to-cost ratio $x_0$ is:

$$P[A_0 \text{ wins } | x_0, c_0] = [1 - F_X(x_0)]^n \tag{31}$$

Hence, the probability that he loses at the same $x_0$ is:

$$P[A_0 \text{ loses } | x_0, c_0] = 1 - [1 - F_X(x_0)]^n \tag{32}$$

This probability also equals the probability that *any* other contractor, besides $A_0$, is the low bidder. Since the remaining $n$ competitors are assumed equivalent, and only one of them can be the low bidder at the same time, their probability of winning equals:

$$\begin{aligned} P[A_i \text{ wins } | x_0, c_0] &= \frac{P[A_0 \text{ loses } | x_0, c_0]}{n} \\ &= \frac{1}{n}\{1 - [1 - F_X(x_0)]^n\} \end{aligned} \tag{33}$$

This result is identical to the one obtained above using the total probability theorem.

To the best of the author's knowledge, equations (29) and/or (33) have not been published before. This fact is probably the major cause of misconceptions about the validity of Friedman's model, as illustrated below.

Continuing the example where $A_0$ chooses $x_0$ to be the median of the bid-to-cost ratio $X$:

$$\begin{aligned} P[A_0 \text{ wins } | x_0 = \text{ median of } X] &= 0.5^n && (34) \\ &= 0.5^4 = 0.0625 \ll 0.2 \\ P[A_i \text{ wins } | x_0 = \text{ median of } X] &= \frac{1 - 0.5^n}{n} && (35) \\ &= \frac{1 - 0.5^4}{4} = 0.234375 > 0.2 \end{aligned}$$

The above probabilities obviously do sum up to one, as required by probability theory, a result whose proof has eluded construction researchers since the late 1960's. The fact that 0.0625 is such a small number simply illustrates the fallacy of bidding at such a high bid-to-cost ratio against four opponents (Rosenshine 1972).

## The Symmetry-Based Errors in Criticizing Friedman's Model

It is interesting to notice that many researchers proclaimed that Friedman's *modeling* assumptions are incorrect since they apparently violated the axioms of probability theory. For example:

> "If all bidders are assumed to be *average*, Friedman's model ... shows five equal competitors who always bid FBC = MBC (MBC is the median of $X$) to each have 0.0625 probability of winning rather than the balanced $1/(n+1) = 1/5 = 0.2$. The winner of the remaining $1 - 5(0.0625) = 0.6875$ projects remains a mystery." (Carr, 1983)

And:

> "Friedman's model does not do this (i.e. produce results supported by symmetry), e.g., in the case in which there are $n$ bidders, and each has the same distribution of bids—regardless of whether this policy is optimal or not for the present—then one

> would expect the probability of a particular bidder winning to be $1/n$. This is clear from arguments of symmetry, and from common sense. By similar arguments, the chance of a bidder, beating any of his competitors will be $1/2$. Using Friedman's expression, the probability of the first bidder beating all of his competitors and winning the contract comes out at $1/2^{(n-1)}$ which is clearly not true." (Dixie 1974)

The errors in these statements should now be obvious. They apply Friedman's model, as derived from the point of view of a competitor, to the case where the state of information and control is that of an outsider.

The product 5(0.0625) equals:

$$5(0.0625) = \sum_{i=0}^{4} P[A_i \text{ wins } | x_i = \text{ median of } X] \tag{36}$$

which is the sum of 5 conditional probabilities each based on a *different* conditional event: "given $x_i$ = median of $X$". For example, the conditional space for $i = 0$ is "given $x_0$ = median of $X$" (which reflects $A_0$'s state of information); the conditional space for $i = 1$ is "given $x_1$ = median of $X$" (which reflects $A_1$'s state of information), etc. As a result, each of the above probabilities is defined in a different conditional space, since the events, "$x_i$ = median of $X$", are not all "given" simultaneously.

Probability theory requires that the sum of the probabilities of mutually exclusive and collectively exhaustive events in the *same* conditional sample space do sum up to one. This is clearly not the case here. In fact, the summation of probabilities from different conditional spaces can also produce a result that is greater than one.

Friedman's model, as shown in equation (10), is non-symmetric because it is formulated from the point of a view of a contractor $A_0$ who knows his cost estimate $c_0$ and controls the value of his bid $b_0$. In considering the case of equal FBC$_i$'s we must require that $c_0$ and hence $b_0$ be unknown. Otherwise, the state of information about the bidding behavior of $A_0$ would not be symmetrical to the state of information about each of the other contractors. Under these conditions, the perspective is that of an outsider, and the corresponding version of Friedman's model is given by expression (28) *and not* (10). As shown earlier, the probability of winning according to equation (28) is indeed symmetric: $1/(n+1)$.

The fact that all FBC$_i$'s have been assumed to equal the median of $X$ does not have any special significance. It is only a special case of the general situation described previously, where all the FBC$_i$'s were assumed equal to some $x_0$. Any value $x_0$ will produce the same result.

If we assume that $c_0$, and hence $b_0$, are known, then we should apply the non-symmetric formulas (10) *and* (29). Expression (10) gives the conditional probability that $A_0$ wins, and expression (29) gives the conditional probability that a particular competitor $A_i$ wins, both from $A_0$'s perspective.

## The Criticism of Friedman's Assumptions

Friedman's model has also been criticized on the following grounds:

> "The fallacy in Friedman's argument arises from the fact that, if it is known that one's bid is lower than that of one of the $n$ competitors, the probability of winning the contract by beating the other $(n-1)$ competitors should be modified." (Dixie 1974)

This criticism of Friedman's model is again unfounded. In this case, the number of competitors $n$ is obviously reduced by one and thus the probability of winning according to expression (9) is indeed modified (i.e. increased).

Furthermore, we must not forget that the estimation of the distribution of the apparent bid-to-cost ratios $F_X(x)$ is based on data (i.e known values $x_j$ from previous projects) and subjective information that is available *prior* to bidding. Obviously, the shape and type of this distribution changes as more data become available.

If we happen to know competitor $A_i$'s bid $b_i$ for a new project, then we would probably decide to change our own bid (especially if $b_i < b_0$). To do so, we must reevaluate the probability of winning at any chosen bid-to-cost ratio $x_0$ (where $x_0 < x_i$), given this new information. At this point we have the freedom to proceed in one of two ways: (a) assume that the distribution $F_X(x)$ is indeed valid and perfectly known, and treat the observed $x_i$ as simply a realization of the random variable $X_i$ with no consequence whatsoever (as long as our bid-to-cost ratio is less than $x_i$), or (b) use the observed $x_i$ as a data point and thus update the shape and the parameters of the distribution $F_X(x)$ for each of the other $n-1$ competitors using Bayesian or classical statistics.

The decision on which of the two approaches, (a) or (b), to follow depends on the strength of our prior belief as to the accuracy of $F_X(x)$ in representing the apparent bid-to-cost ratios for this project, and the likelihood (in a Bayesian sense) of observing the sample point $x_i$. In the extreme, it is in fact possible (from a Bayesian point of view) to produce a subjective estimate of $F_X(x)$ based on the single observation $x_i$, that carries so much weight that we practically discard all the information contained in the data $x_j$ from previous projects. This is a statistical decision that is up to the individual contractor.

What is really criticized in the above statement is Friedman's assumption that the random variables $X_i$ can be considered mutually independent. Even though the modeling validity of this assumption is certainly debatable, it is important to notice, however, that unless it is accepted the observed data from previous projects will be *correlated*. Furthermore, the estimation of the marginal distribution $F_X(x)$ is hardly sufficient under these conditions.

If the $X_i$ are not independent, then the distribution of $X_{min}$ requires the estimation of the shape and parameters of the joint cumulative distribution function $F_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n)$ for *any* possible number of competitors $n$. Given the practical difficulties of this task, and given that the only data available from past projects are the contractor's estimate and the bids of his opponents, it is practically impossible to reject Friedman's assumption that the $X_i$ are mutually independent.

The difficulty of this task is best illustrated by the following contradiction: After rejecting Friedman's second assumption of independence, the same author proposes a "correct" procedure for computing the probability to win, which is offered as an alternative to Friedman's model, and which "does not" require this assumption. In illustrating this "correct" procedure, this author bypasses the problems of dependence among the $X_i$'s, by implicitly assuming that the unknown ratios of each competitor's bid $B_i$ to the unknown (*a priori*) actual cost of the project (as if though the *true* cost of the project does not depend on which contractor actually does the work) are mutually independent (Dixie 1974).

## Conclusion

As a general observation it should be noted that arguments concerning the validity of a model based on mathematical inconsistency should be carefully scrutinized since they may be in error. Any set of modeling assumptions that does not violate the axioms of probability theory should always produce results (theorems) that are also compatible with the axioms. An incompatibility with the axioms automatically implies a mathematical — not a modeling — error: either the axioms are inappropriate or the results (theorems) are incorrect.

Irrespective of whether Friedman's assumptions of independence provide an accurate model of competitive bidding in the real world, they certainly do not violate the axioms of probability theory. Based on this argument alone, his model should have been more closely examined and better understood before the verdict of probabilistic inconsistency was pronounced.

The fact that Friedman's model yields different results, depending on whether it is based on the state of information and control of a competitor, as opposed to that of an outsider, is a characteristic that should be *a priori* required of any bidding model. A competitor is part of the game, has full control of his bid, and can thus affect the outcome of the process. An outsider simply observes, has limited information, and cannot control any bids or the final outcome. Any bidding model that does not differentiate between these two perspectives is certainly unacceptable. It *dilutes* the information available to a competitor (the real user of the model) and produces probabilities of winning that tend to be higher when they should have been very low and vice versa.

## Appendix I. — References

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## Appendix II. — Notation

*The following symbols are used in this paper:*

$A_i$ = The $i$th competing contractor.

AFBC$_i$ = $X_i = B_i/c_0$. Contractor $A_i$'s apparent bid-to-cost ratio (as viewed by $A_0$).

$B_i$ = The bid of contractor $A_i$.

$C_i$ = The project cost estimate for contractor $A_i$.

FBC$_i$ = $B_i/C_i$. Contractor $A_i$'s (true) bid-to-cost ratio.

$F_X(x)$ = $P[X < x]$. The cumulative distribution function of the random variable $X$ evaluated at value $x$.

$F_{X|X_0=x_0}(x)$ = $P[X < x|X_0 = x_0]$. The conditional cumulative distribution function of the random variable $X$, given that $X_0 = x_0$, evaluated at value $x$.

$f_X(x)$ = $P[x < X < x + dx]/dx$. The probability density function of the random variable $X$ evaluated at value $x$.

$f_{X|X_0=x_0}(x)$ = $P[x < X < x + dx|X_0 = x_0]/dx$. The conditional probability density function of the random variable $X$, given that $X_0 = x_0$, evaluated at value $x$.

$m_0$ = The markup of contractor $A_0$.

$n$ = The number of competitors bidding against contractor $A_0$.

$X_i$ = $B_i/c_0$. Contractor $A_i$'s apparent bid-to-cost ratio (as viewed by $A_0$).

$X_{min}$ = The minimum apparent bid-to-cost ratio of the $n$ opponents of $A_0$.