CENTER FOR CONSTRUCTION ENGINEERING AND MANAGEMENT

Optimal Capital Structure for Privately-Financed Infrastructure Projects

Valuation of Debt, Equity and Guarantees

Antonio Dias, Jr. and Photios G. Ioannou

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Forward

Concession agreements can be used by governments to induce the private sector to develop and operate many types of infrastructure projects. Under this type of arrangement, several private-sector companies join forces, become project promoters, and form a separate company that becomes responsible for financing, building, and operating the facility. Before this company can be formed, prospective promoters must determine how to fund the associated construction and startup costs. They must decide how much to borrow, how much to infuse from their own funds and how much to raise from outside investors. Typically, such projects must repay any debt obligations through their own net operating income, and do not provide the lenders with any other collateral (off-balance-sheet financing). Thus, the possibility of a costly bankruptcy becomes much more likely.

A mathematical formulation for determining the value of debt and equity as well as the optimal financial structure for privately-promoted projects is presented here. It is shown that under off-balance-sheet financing and the possibility of a costly bankruptcy, the maximum amount of debt that a project can service (its debt capacity) is less than 100% debt financing. Furthermore, the amount of debt that maximizes the promoters' return on equity is always less than the project's debt capacity and the amount of debt that maximizes the project's net present value is even smaller. Exceeding these debt amounts and moving towards debt capacity should be avoided as it can rapidly erode the project's value to the investors. Finally, it is demonstrated that both debt levels as well as the promoters' return increase through the provision of either production or minimum-revenue guarantees.

A multiattribute additive hierarchical model, called the Desirability Model, has been developed to evaluate (i) the capability of companies to participate in the promotion of projects and (ii) the feasibility of projects to be pursued by private promotion.

Questionnaires were sent to 15 renowned experts to gather information about the desirable attributes of promoting companies and projects. A total of 23 attributes have been identified as able to characterize the quality level of companies and projects. Validation was performed and the results indicate that the model closely captures the preferences of the respondents. UMCCE Report No. 95-09 presents the Desirability Model in detail and provides a through discussion of the essential issues and concepts involved in the promotion of projects via concession arrangements or privatization.

The contents of this report and of UMCEE Report No. 95-09 were originally published by Dias (1994).

This has proven to be an exciting field of study and continuous research are being performed by the authors. For further information on this report or on our on-going research materials contact us at the following addresses:

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Chapter 1

Debt Capacity and Optimal Capital Structure for Privately-Financed Infrastructure Projects

''An 'engineer' is one who can do with a dollar what any bungler can do with two.''

Arthur Wellington, 1887

1.1 Introduction

As existing infrastructure ages and demand for new facilities increases due to population growth and technological advancement, governments worldwide no longer have funds in place, or the bonding capacity required, to finance all the public facilities, public services, and infrastructure that they would like to provide. In the US, for example, the availability of federal grants for public works projects has been constrained by budget deficits, while the ability of state and municipal governments to finance construction through bond issues has been affected by changes in tax laws and limits on debt capacity imposed by law, political considerations, or capital markets

(Beidleman, 1991). According to Aschauer (1991), the lack of funds to finance infrastructure projects is one of the major causes of the economy's faltering productivity, profitability, and private sector capital formation. He estimated, for example, that a 1% increase in the stock of infrastructure capital would raise American productivity by 0.24%.

Apart from the lack of funding resources, there is an increased understanding on the part of some governments that they should not own and/or operate certain types of facilities and infrastructure because of their less effective utilization of resources, when compared with the more flexible and cost conscious private sector, and because of changes in their political ideologies. Private enterprise can benefit from this situation by providing its financial resources and managerial skills to increase its share of the infrastructure market.

In this report we describe an arrangement for the private financing of infrastructure projects based on concession agreements. In this context, the objective of the report is to illustrate how to determine the debt capacity and optimum financial structure for privately-financed infrastructure projects. This decision is of paramount importance because it constrains the ability of the promoting team to go ahead with the project. If the promoting team does not have the necessary equity to achieve the optimal debt-to-equity ratio, then it should search for additional investors until there are enough resources to achieve the optimal capital structure. A promoting team should not try to borrow as much as it can as this would make it worse off. Furthermore, the determination of debt capacity and optimum financial structure provides the basis for the structure and evaluation of the possible types of guarantees (minimum production, minimum revenue, etc.) that the host government may extend to the project (Dias 1994).

To be as realistic as possible in the evaluation of risky debt, the formulation developed here explicitly considers both the possibility of bankruptcy and the effect of taxes. These are the two main factors that influence debt policy (Brealey and Myers 1991). The explicit consideration of bankruptcy costs is of particular importance because privately-financed projects provide no collateral to debtholders. The discussion begins by using a market equilibrium approach to determine the value of the project and of its financial components (debt and equity). Next, we show that when risky debt exists (*i.e.*, when the cost of bankruptcy is greater than zero), there is a limit on the amount that can be borrowed to fund a project (*i.e.*, the project's debt capacity). Finally, we determine the capital structure that maximizes either the investors' return on equity, or the project's *NPV*, and show that these strategies always require debt levels that are less than the project's debt capacity. The application of these concepts is illustrated with an example.

The nature of this important topic requires a relatively complex mathematical treatment. We have chosen to present and explain the most basic analytical results in some detail so that they could be verified. They may also provide a starting point for further investigation by other researchers. Care has been taken to explain most of the results using common-sense concepts so that even if most of the analysis is ignored, the assumptions, results and especially the conclusions can be understood by a wide audience.

1.2 Concession Agreements

A concession contract is one possible arrangement governments can use to raise the necessary funds to finance revenue-generating projects when their access to traditional sources of capital is constrained or undesirable. Examples of projects that can be funded using concession arrangements include roads, bridges, tunnels, power plants, pipelines, industrial plants, and office buildings. This type of arrangement requires the involvement of several companies (the promoting team) to finance the project, perform the design, execute and manage its construction, and be responsible for the operation and maintenance of the facility. Depending on the nature of the project, the promoting team might include construction companies, engineering firms, equipment and material suppliers, plant operators, utility companies, and customers of the facility. Figure 1-1 illustrates possible contractual relationships (dashed lines) and flows of capital (solid lines) among the different participants of a concession-financed project. The shaded boxes indicate those participants that can either be part of the promoting team or serve as external providers of services.

The amount of time promoters have to construct, operate and maintain a facility before transferring its ownership to the project sponsor (usually the government) is known as the concession period. Projects that have finite concession periods are called BOT (Build-Operate-Transfer) projects, otherwise they are called BOO (Build-Operate-Own) projects.

In Build-Operate-Transfer (BOT) projects, the sponsor provides a concession that permits a promoting team to build a facility and to operate it for a specific amount of time. Project

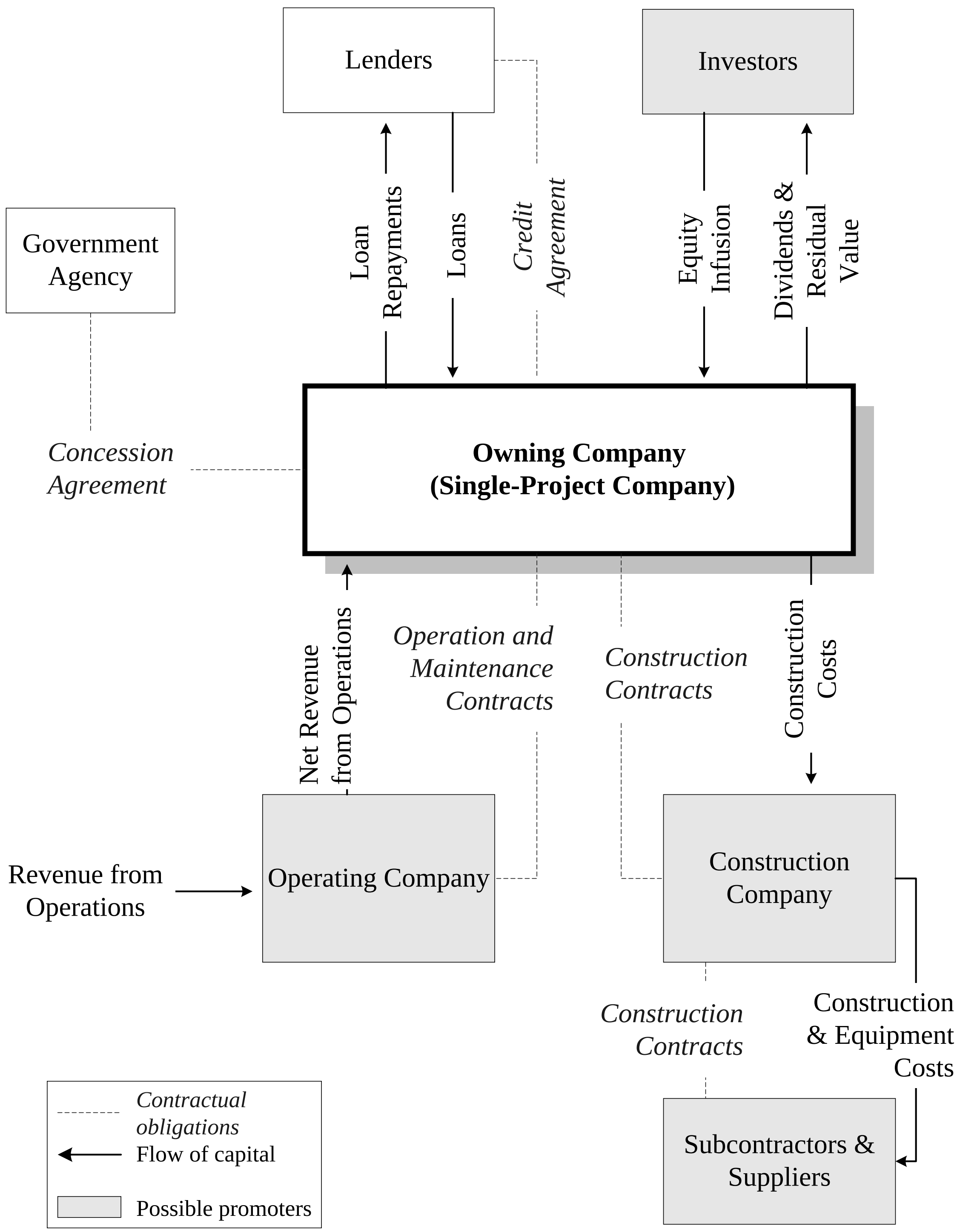

Figure 1-1: Contractual and Financial Structure of a Privately-Financed Project

promoters use the revenues produced during the concession period to pay back lenders, other shareholders, and to get a return on their investment. After the concession period has elapsed, the operation of the facility and its revenues are transferred to the sponsor that infused, at the time of construction, very few monetary resources. One very well publicized example of this method is the Channel Tunnel project linking France and the UK by rail. Build-Operate-Own (BOO) projects should also produce revenues from their cash flows to cover debt, operation and maintenance costs and to return profit gains to promoter companies. However, project promoters have an unlimited amount of time to operate the facility as well as full ownership of the underlying assets. Actual examples of such projects are power plants (constructed and operated by private utility companies) and public office buildings. This process can be used not only for financing but also for the privatization of public services.

Concession-financed projects are funded through a combination of debt and equity capital. Debt is provided by lending institutions (*e.g.*, banks) while equity is provided by the companies that have an interest in the project (*i.e.*, the promoting companies) and by companies that view the project as an investment opportunity (*e.g.*, pension funds). The use of debt is essential to fund large infrastructure concession projects because promoters rarely have all the necessary financial resources. However, the use of equity is also essential as it complements debt financing and more easily accommodates the financial needs of the project. (Debt instruments present rigid payment dates and amounts and do not normally offer large grace periods. Equity is more flexible as dividends are paid based on the availability of funds.)

Once a project concession is granted by the sponsor, the promoting team creates a company, referred to as the ‘‘owning company,’’ which is responsible for the financing, construction, and operation of the facility and which retains ownership during the concession period. The creation of an owning company as a separate entity is of great benefit to the promoting companies because it allows them to raise debt without providing a portion of their own assets as collateral. That is, the revenues of the project are the only source to repay the debt. In the case where the project does not produce enough revenues to fully repay the debt, the lenders receive only a partial payment of the debt obligations and do not have any rights to demand full payment from the promoters. This type of financing is known as off-balance-sheet financing. The debt raised to fund the project is not secured by the promoters, and hence it does not appear on their balance-sheets, but only on the balance-sheet of the owning company.

1.3 Project Valuation Using the CAPM

No finance theory can give a satisfactory explanation of the valuation of a firm if it fails to take into account the equilibrium of capital markets. The Capital Asset Price Model (CAPM), developed by Sharpe (1964), Lintner (1965), and Mossin (1966) is one such theory. It shows that the equilibrium rate of return on an asset is a function of its relative risk level when compared to the market portfolio. The market portfolio consists of a weighted average of all assets on the market; that is, each asset contributes to the portfolio by the proportion of its value to the total market value of the assets. The essential relationship of the CAPM is:

$$E[\tilde{r}_i] = r_f + \beta_i \left(E[\tilde{r}_m] - r_f\right) \tag{1.1}$$

Note that throughout this report random variables are indicated by placing a tilde ($\sim$) over their names. The CAPM indicates that, if the market is in equilibrium at time $t-1$, $E[\tilde{r}_i]$, the expected return on a risky asset (*e.g.*, a project) i during the period $(t-1, t)$, is, r_f, the risk-free rate of interest during that period plus a risk premium, which is determined by $E[\tilde{r}_m] - r_f$, the excess rate of return on the market portfolio (above the risk-free rate) and β_i, the systematic risk of asset i. Systematic risk, also called market risk, exists because there are economy-wide factors that affect the entire market and cannot be avoided no matter how much diversified a portfolio of assets is. It is measured by determining the sensitivity of the returns on asset i to market movements, that is:

$$\beta_i = \frac{\mathrm{Cov}(\tilde{r}_i, \tilde{r}_m)}{\sigma_m^2} \tag{1.2}$$

An asset that has $\beta > 1$ is more sensitive to market movements than the market portfolio, and thus more risky, and should provide returns greater than the expected return on the market portfolio. Similarly, an asset with $\beta < 1$ is less risky than the market portfolio. The derivation of the CAPM, as well as its underlying assumptions, appear in Copeland (1988, pp.195-198).

Hamada (1971) notes that, in a single-period situation, the CAPM relationship (Eq. 1.1) can be viewed not only as the market equilibrium relationship between the expected rate of return on asset i and its individual risk, but also as a minimum expected rate of return required by the market for a given level of systematic risk. Thus, it provides a cut-off rate against which the expected rate of return on project i can be compared. For example, if the cost of investing in project i is A_i, its expected value at the end-of-period is $E[\tilde{V}_{i,1}]$, and its systematic risk is β_i then, in order to be accepted (*i.e.*, to have a positive net present value) the project must satisfy the following condition:

$$\frac{E[\tilde{V}_{i,1}]}{1+E[\tilde{r}_i]} = \frac{E[\tilde{V}_{i,1}]}{1+r_f+\beta_i\left(E[\tilde{r}_m]-r_f\right)} \geq A_i \tag{1.3}$$

Based on Hamada's interpretation, the CAPM can be used to determine the present value of a project when the market is in equilibrium. To see this let us define the following rates of return:

$$\tilde{R} = 1+\tilde{r} \text{ (one plus the rate of return on a single-period project),} \tag{1.4}$$

$$\tilde{R}_m = 1+\tilde{r}_m \text{ (one plus the rate of return on the market), and} \tag{1.5}$$

$$R_f = 1+r_f \text{ (one plus the risk-free rate of interest)} \tag{1.6}$$

Note that throughout this report we show variables that represent one-plus-the-rate-of-return with capital letters (*e.g.*$\tilde{R}_i = 1+\tilde{r}_i$ and $\widetilde{ROE} = 1+\widetilde{Roe}$). Given this convention, (1.1) can be expressed as:

$$E[\tilde{R}] = R_f + \frac{\text{Cov}(\tilde{R},\tilde{R}_m)}{\sigma_m^2}\left(E[\tilde{R}_m]-R_f\right) \tag{1.7}$$

By definition, we have:

$$\tilde{R} = \frac{\tilde{V}_1}{V} \tag{1.8}$$

$$\text{Cov}(\tilde{R},\tilde{R}_m) = \text{Cov}(\frac{\tilde{V}_1}{V},\tilde{R}_m) = \frac{1}{V}\,\text{Cov}(\tilde{V}_1,\tilde{R}_m) \tag{1.9}$$

where V is the present (actual) market value of the project when the market is in equilibrium and $\tilde{V}_1$ is the uncertain end-of-period value of the project. Substituting (1.8) and (1.9) into (1.7) gives:

$$\frac{E[\tilde{V}_1]}{V} = R_f + \left(E[\tilde{R}_m]-R_f\right)\frac{\text{Cov}(\tilde{V}_1,\tilde{R}_m)}{\sigma_m^2 V} \tag{1.10}$$

and rearranging the terms:

$$V = \frac{E[\tilde{V}_1] - \frac{E[\tilde{R}_m]-R_f}{\sigma_m^2}\,\text{Cov}(\tilde{V}_1,\tilde{R}_m)}{R_f} = \frac{E[\tilde{V}_1]-\lambda\,\text{Cov}(\tilde{V}_1,\tilde{R}_m)}{R_f} \tag{1.11}$$

where λ is the market price of a unit of risk. Note that $E[\tilde{V}_1]-\lambda\,\text{Cov}(\tilde{V}_1,\tilde{R}_m)$ is the certainty equivalent (as determined by the market) of the end-of-period value of project $\tilde{V}_1$, and that is why it is discounted by the risk-free rate (instead of $\tilde{R}$) in order to calculate the actual market value of the project.

1.3.1 Bankruptcy Costs

Let us consider a one-period privately-financed project, that costs a certain amount A to be built, is financed through the use of equity and debt, and generates a net operating income $\tilde{X}$ at the end of its operational period. Then, the end-of-period market value of the project, $\tilde{V}_1$, can be calculated by summing the end-of-period market values of the outstanding debt and equity:

$$\tilde{V}_1 = \tilde{D}_1 + \tilde{S}_1 \tag{1.12}$$

The end-of-period market value of equity, $\tilde{S}_1$, is uncertain as the earnings received by the equityholders depend on the net operating income of the project, $\tilde{X}$, and on the the amount of debt outstanding. The end-of-period market value of debt, $\tilde{D}_1$, is uncertain because it also depends on the net operating income of the project and because the debt repayment is not guaranteed by the promoting companies. If the net operating income, $\tilde{X}$, is greater than the amount borrowed at the beginning of the project (debt principal) plus the promised interest, then the debtholders will receive the full promised amount d_1 (principal plus interest) at the end of the period. On the other hand, if the project does not produce a net operating income sufficient to repay the debt ($\tilde{X} < d_1$), the owning company does not meet its debt obligation, enters a state of financial distress and becomes bankrupt. In this case, debtholders take ownership of the company and pay the costs of bankruptcy before they receive any payment. Details of this process are described in Martin and Scott (1976), Hong and Rappaport (1978), and Kim (1978).

Kim (1978) discusses the different types of bankruptcy costs and classifies them into two categories: direct costs and indirect costs. For infrastructure projects, direct costs include administrative expenses (*e.g.*, legal fees, trustee fees, referee fees, and time lost by executives in litigation). Indirect costs are incurred basically in the form of trustee certificates. These certificates are used to raise new capital for the continuance of the services provided by the project facility and become senior instruments to the outstanding debt of the bankrupt company. In this report, bankruptcy costs are represented by the following linear function (Kim 1978):

$$\tilde{B} = b_f + b_v\tilde{X} \quad (0 \leq \tilde{B} \leq \tilde{X}) \tag{1.13}$$

where $\tilde{B}$ represents the uncertain cost of bankruptcy and is a positive-non-greater-than function of $\tilde{X}$; b_f represents the expected value of the components of bankruptcy costs (expressed in monetary units) that are independent of the company's net operating income $\tilde{X}$ (*i.e.*, those costs that do not depend on the size of the owning company); and b_v is a variable cost coefficient that can assume values from -1 to 1 and which relates the costs of bankruptcy, $\tilde{B}$, to the net operating income $\tilde{X}$ (if $\tilde{B}$ is independent of $\tilde{X}$ then b_v is zero).

For convenience, it is assumed that once in bankruptcy the owning company is liquidated and its proceedings get distributed according to the Bankruptcy Reform Act of 1978. Thus, administrative expenses associated with liquidating the project (*i.e.*, bankruptcy costs), such as fees and other compensation paid to trustees, attorneys, accountants, etc., are paid before the debtholders claims on the project assets. Empirical studies on bankruptcies show that administrative expenses range from 4 to 20% of a company's assets depending on the type of company analyzed and other sample characteristics (Van Horne, 1986).

1.3.2 The Present Value of Debt

For a one-period privately-financed project, the amount D that the owning company can borrow depends on the risk characteristics of the amount d_1 it promises to repay at the end of the period,

$$d_1 = D(1 + \mathit{Int}) \tag{1.14}$$

where *Int* is the nominal interest rate charged by the lenders. Because the loan is risky, however, the amount $E[\tilde{D}_1]$ that the lenders expect to receive is less than the full promised amount d_1 and their expected return $E[\tilde{r}_D]$ is less than *Int*:

$$E[\tilde{D}_1] = D(1 + E[\tilde{r}_D]) \tag{1.15}$$

For the same reason, as the promised amount d_1 increases, so does the risk faced by the debtholders and so does the nominal interest rate *Int* they demand. As shown below, when d_1 reaches a certain level, the required nominal interest *Int* is so large that the debt amount D can actually decrease (even though the owning company promises to pay more). To illustrate this behavior in a mathematically tractable manner, and without loss of generality, the remaining discussion focuses on the evaluation of debt and equity for a one-period project. The analysis for multiperiod projects, although similar, is best undertaken using numerical methods.

The loan amount D (*i.e.*, the present value of a project's debt as determined by the market) can be computed by following the same line of reasoning used to determine V, the present market value of a project (Eq. 1.11),

$$D = \frac{E[\tilde{D}_1] - \lambda \,\text{Cov}(\tilde{D}_1, \tilde{R}_m)}{R_f} \tag{1.16}$$

where $E[\tilde{D}_1]$ is the expected value of the debt at time 1; λ is the market price per unit of risk; Cov$(\tilde{D}_1, \tilde{R}_m)$ is the covariance between the value of debt at time 1 and one-plus-the-rate-of-return-on-the-market ; and R_f is one-plus-the-risk-free-rate.

The end-of-period value of debt, $\tilde{D}_1$, depends on the end-of-period project net operating income, $\tilde{X}$, and can be expressed as:

$$\tilde{D}_1 = \begin{cases} d_1 & \text{if } \tilde{X} \geq d_1 \\ \tilde{X} - \tilde{B} & \text{if } \tilde{B} \leq \tilde{X} < d_1 \\ 0 & \text{if } \tilde{X} < \tilde{B}, \textit{ i.e.}, \tilde{X} < b' = \frac{b_f}{1-b_v} \end{cases} \tag{1.17}$$

Thus, if the net operating income at the end of the period is greater than the promised amount d_1, the debtholders receive the full debt payments. Otherwise, they receive the net operating income minus the bankruptcy costs, provided this difference is positive, and nothing if the difference is negative (the entire net operating income is consumed by bankruptcy costs). Alternatively, $\tilde{D}_1$ can be expressed in the following equation form:

$$\tilde{D}_1 = d_1(1-\delta_b)\delta_q + \delta_b\delta_q\tilde{X} - \delta_b\delta_q(b_f + b_v\tilde{X}) \tag{1.18}$$

where δ_b and δ_q are binary variables defined as follows:

$$\delta_b = \begin{cases} 0 & \text{if } \tilde{X} \geq d_1 \\ 1 & \text{if } \tilde{X} < d_1 \end{cases} \tag{1.19}$$

$$\delta_q = \begin{cases} 0 & \text{if } \tilde{X} < b' = \frac{b_f}{1-b_v} \\ 1 & \text{if } \tilde{X} \geq b' \end{cases} \tag{1.20}$$

Note that when the owning company is not bankrupt, *i.e.*, $\tilde{X} \geq d_1$, then $\delta_b = 0$, $\delta_q = 1$, and $\tilde{D}_1 = d_1$. Similarly, when in bankruptcy and $\tilde{X} \geq b'$, *i.e.*, $\tilde{B} \leq \tilde{X} < d_1$, then $\delta_b = 1$, $\delta_q = 1$, and $\tilde{D}_1 = (1-b_v)\tilde{X} - b_f$. Finally, when in bankruptcy and $\tilde{X} < b'$, *i.e.*, $\tilde{X} < \tilde{B}$, then $\delta_b = 1$, $\delta_q = 0$, and $\tilde{D}_1 = 0$. Figure 1-2 shows the value of $\tilde{D}_1$ as a function of $\tilde{X}$. The graph on the left, shows $\tilde{D}_1$ (when bankruptcy costs are not considered) and $\tilde{B}$ (without the restriction $\tilde{B} \leq \tilde{X}$). The

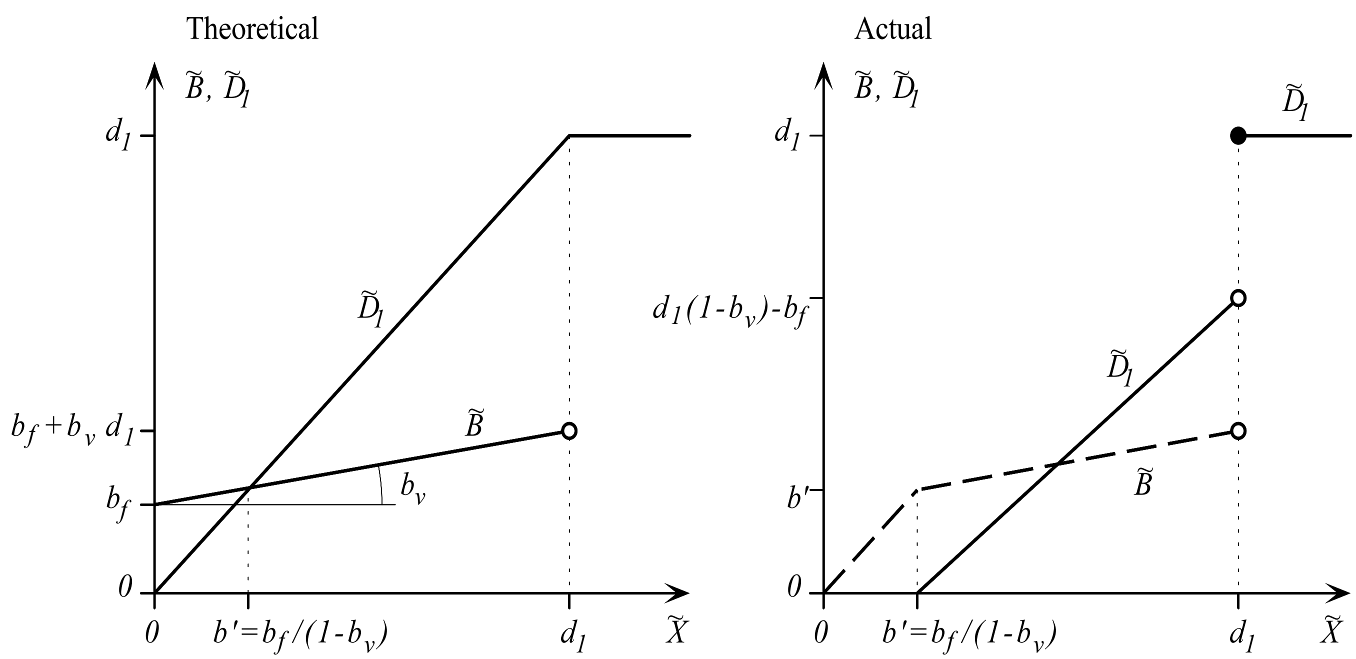

Figure 1-2: Value of Debt ($\tilde{D}_1$) and Bankruptcy Costs ($\tilde{B}$) as a Function of the NOI ($\tilde{X}$)

graph on the right, illustrates the actual values of $\tilde{D}_1$ given by (1.18), when bankruptcy costs are considered, and the values of $\tilde{B}$ with the restriction $\tilde{B} \leq \tilde{X}$ imposed.

The expected value of the end-of-period debt, $E[\tilde{D}_1]$, can be calculated from (1.18) as:

$$E[\tilde{D}_1] = d_1 \left(E[\delta_q] - E[\delta_b \delta_q]\right) + (1 - b_v) E[\delta_b \delta_q \tilde{X}] - b_f E[\delta_b \delta_q] \tag{1.21}$$

The expected values on the right-hand side of (1.21) are derived in Appendix A under the assumption that $\tilde{X}$ follows a Normal distribution. Substituting (A.12), (A.13), and (A.15) into (1.21) and rearranging the terms gives:

$$\begin{aligned} E[\tilde{D}_1] &= d_1 \left(1 - F_X(d_1)\right) + (1 - b_v) \left\{ E[\tilde{X}] \left(F_X(d_1) - F_X(b')\right) + \sigma_X^2 \left(f_X(b') - f_X(d_1)\right) \right\} \\ &\quad - b_f \left(F_X(d_1) - F_X(b')\right) \end{aligned} \tag{1.22}$$

Therefore, the expected end-of-period payment to debtholders after bankruptcy costs (*i.e.*, the value of debt at time 1) is the full promised amount d_1 multiplied by the probability that the project does not go bankrupt plus the conditional expected end-of-period project net cash flow given that the project is bankrupt minus the expected value of the bankruptcy costs.

The $\text{Cov}(\tilde{D}_1, \tilde{R}_m)$ in (1.16) can be expressed as:

$$\text{Cov}(\tilde{D}_1, \tilde{R}_m) = \text{Cov}(d_1(1 - \delta_b)\delta_q + \delta_b \delta_q \tilde{X} - \delta_b \delta_q (b_f + b_v \tilde{X}), \tilde{R}_m)$$

$$= \quad d_1 \text{ Cov}(\delta_q, \tilde{R}_m) - (d_1 + b_f) \text{ Cov}(\delta_b \delta_q, \tilde{R}_m) + (1 - b_v) \text{ Cov}(\delta_b \delta_q \tilde{X}, \tilde{R}_m) \quad (1.23)$$

Substituting the covariances on the right-hand side of the above equation by (A.21) and (A.23) (Appendix A), rearranging terms, and multiplying both sides by λ gives:

$$\lambda \text{ Cov}(\tilde{D}_1, \tilde{R}_m) = \{(1 - b_v)(F_X(d_1) - F_X(b')) + (b_f + b_v d_1) f_X(d_1)\} \lambda \text{ Cov}(\tilde{X}, \tilde{R}_m) \quad (1.24)$$

Eq. 1.24 shows that the systematic risk premium on the project's debt, λ Cov$(\tilde{D}_1, \tilde{R}_m)$, is equal to the project's systematic operating risk premium, λ Cov$(\tilde{X}, \tilde{R}_m)$, multiplied by a factor that represents the probability that debtholders would only receive some partial payment (given by the occurrence of bankruptcy) plus the systematic risk premium on the project's bankruptcy costs, λ Cov$(\tilde{X}, \tilde{R}_m)[(b_f + b_v d_1) f_X(d_1) - b_v(F_X(d_1) - F_X(b'))]$.

Therefore, if the company is not bankrupt at the end of the period (or if the company has a 0% probability of going into bankruptcy), debtholders receive a fixed amount d_1 that has no systematic relationship with the market. It is only when the company is bankrupt that $\tilde{D}_1$ (and also $\tilde{X}$) presents a systematic risk that cannot be diversified away by the debtholders. This is reflected in the following equation for β_D:

$$\begin{aligned} \beta_D &= \frac{\text{Cov}(\tilde{R}_D, \tilde{R}_m)}{\sigma_m^2} = \frac{\text{Cov}(\tilde{D}_1, \tilde{R}_m)}{D\sigma_m^2} \\ &= \frac{\text{Cov}(\tilde{X}, \tilde{R}_m)}{D\sigma_m^2} \{(1 - b_v)(F_X(d_1) - F_X(b')) + (b_f + b_v d_1) f_X(d_1)\} \end{aligned} \quad (1.25)$$

From (1.25) one can see that β_D increases as bankruptcy costs increase. This can be shown by examining the terms in (1.25) that depend on b_f and b_v. Given that $d_1 \leq E[\tilde{X}]$, the area of a rectangle with base d_1 and height $f_X(d_1)$ is always greater than the area under a Normal distribution $F_X(d_1) - F_X(b')$. Thus, it follows that $(b_f + b_v d_1) f_X(d_1)$ is always greater than $b_v(F_X(d_1) - F_X(b'))$. This relationship together with the condition Cov$(\tilde{X}, \tilde{R}_m) > 0$ implies that higher bankruptcy costs result in a higher systematic risk premium on the project's debt. In other words, as bankruptcy costs increase so does the risk premium required by the debtholders.

The market value of debt, D, can be calculated by substituting (1.22) and (1.24) into (1.16):

$$\begin{aligned} D = \; & \frac{1}{R_f} \Big\{ d_1 (1 - F_X(d_1)) + (1 - b_v) \Big\{ E[\tilde{X}] (F_X(d_1) - F_X(b')) + \sigma_X^2 (f_X(b') - f_X(d_1)) \Big\} - \\ & - b_f (F_X(d_1) - F_X(b')) - \lambda \text{ Cov}(\tilde{X}, \tilde{R}_m) [(1 - b_v)(F_X(d_1) - F_X(b')) + \\ & + (b_f + b_v d_1) f_X(d_1)] \Big\} \end{aligned} \quad (1.26)$$

Note that, in the absence of bankruptcy costs (*i.e.*, $b_f = b_v = 0$), the present value of debt (*i.e.*, the amount that the owning company can borrow) is the promised amount d_1 to be paid to debtholders at the end of the period multiplied by the probability that the company does not go bankrupt plus the conditional expected value of the project's net income given that the company goes into bankruptcy minus the company's operating risk premium, λ Cov$(\tilde{X}, \tilde{R}_m)$, multiplied by the company's probability of bankruptcy (all discounted at the risk-free rate).

1.3.3 The Present Value of Equity

The amount of money that must be invested as equity in a project that costs A, given a debt level D, is $A - D$. As soon as this money is invested in a project with a positive *NPV*, however, its market value increases to S, and the net present value of the project is

$$\textit{NPV} = S - (A - D) = S + D - A = V - A \tag{1.27}$$

The present (market) value of S can be expressed in the same form used to express the actual market value of the debt of a project, D, that is

$$S = \frac{E[\tilde{S}_1] - \lambda \text{ Cov}(\tilde{S}_1, \tilde{R}_m)}{R_f} \tag{1.28}$$

where $E[\tilde{S}_1]$ is the expected value of the end-of-period value of equity and Cov$(\tilde{S}_1, \tilde{R}_m)$ is the covariance between the end-of- period value of equity and the return on the market.

The end-of-period value of equity, $\tilde{S}_1$, represents the market value of the company after debt obligations are paid to debtholders and taxes are paid to the government, that is:

$$\tilde{S}_1 = (1 - T)(\tilde{X} - \textit{DInt} - \textit{Dep}) + \textit{Dep} - D \tag{1.29}$$

where *DInt* is the interest due on the debt and *Dep* is depreciation. As $D(1 + \textit{Int}) = d_1$ and assuming that $\textit{Dep} = A$, (1.29) can be rewritten as:

$$\tilde{S}_1 = (1 - T)(\tilde{X} - d_1) + T(A - D) \tag{1.30}$$

Equityholders have limited liability and do not have any obligations to pay if the company goes bankrupt (*i.e.*, if $\tilde{X} < d_1$ then $\tilde{S}_1 = 0$). As a result, $\tilde{S}_1$ can be expressed more accurately as:

$$\tilde{S}_1 = \begin{cases} (1 - T)(\tilde{X} - d_1) + T(A - D) & \text{if } \tilde{X} \geq d_1 \\ 0 & \text{if } \tilde{X} < d_1 \end{cases} \tag{1.31}$$

Alternatively, $\tilde{S}_1$ can be expressed in the following equation form:

$$\tilde{S}_1 = (1-T)[\tilde{X} - d_1](1-\delta_b) + T(A-D)(1-\delta_b) \tag{1.32}$$

Using the relationships developed in Appendix A, the expected value of the end-of-period equity, $E[\tilde{S}_1]$, is:

$$E[\tilde{S}_1] = \left[(1-T)(E[\tilde{X}] - d_1) + T(A-D)\right](1 - F_X(d_1)) + (1-T)\sigma_X^2 f_X(d_1) \tag{1.33}$$

Therefore, the expected end-of-period value of the owning company after all obligations have been satisfied is the after-tax conditional expected value of the project's net operating income given that the company is not bankrupt, $(1-T)[E[\tilde{X}](1-F_X(d_1)) + \sigma_X^2 f_X(d_1)]$, minus the after-tax value of the debt obligations multiplied by the probability that the company is not bankrupt, $(1-T)d_1(1-F_X(d_1))$, plus the expected value of the tax credits, $T(A-D)(1-F_X(d_1))$.

Following the procedure used to calculate $\text{Cov}(\tilde{D}_1, \tilde{R}_m)$, the systematic risk premium on the company's equity, $\lambda\ \text{Cov}(\tilde{S}_1, \tilde{R}_m)$, is given by:

$$\lambda\ \text{Cov}(\tilde{S}_1, \tilde{R}_m) \quad = \quad \{(1-T)(1-F_X(d_1)) + T(A-D)f_X(d_1)\}\ \lambda\ \text{Cov}(\tilde{X}, \tilde{R}_m) \tag{1.34}$$

Here, $\lambda\ \text{Cov}(\tilde{S}_1, \tilde{R}_m)$ is equal to the after-tax project's systematic operating risk premium, $(1-T)\lambda\ \text{Cov}(\tilde{X}, \tilde{R}_m)$, multiplied by the probability that the company does not go bankrupt plus the systematic risk premium on tax credits, $T(A-D)f_X(d_1)\lambda\ \text{Cov}(\tilde{X}, \tilde{R}_m)$.

Finally, the present (market) value of equity, S, can be calculated by substituting (1.33) and (1.34) into (1.28):

$$\begin{aligned} S \quad = \quad & \frac{1}{R_f}\Big\{\left[(1-T)(E[\tilde{X}] - d_1) + T(A-D)\right](1-F_X(d_1)) + (1-T)\sigma_X^2 f_X(d_1) \\ & -\lambda\ \text{Cov}(\tilde{X}, \tilde{R}_m)\left[(1-T)(1-F_X(d_1)) + T(A-D)f_X(d_1)\right]\Big\} \end{aligned} \tag{1.35}$$

1.4 Project Debt Capacity

Project debt capacity, D^c, is defined as the maximum amount an owning company can borrow in a perfect capital market in order to fund a project. For the concept of debt capacity to be meaningful, it is necessary to show that there exists a finite value d_1^c, that satisfies the following two conditions: $\partial D/\partial d_1 = 0$ and $\partial^2 D/\partial d_1^2 < 0$. The first derivative is given by differentiating equation (1.26) with respect to d_1:

$$\begin{aligned} \frac{\partial D}{\partial d_1} &= \frac{1}{R_f}\left\{\frac{\partial d_1}{\partial d_1} - \frac{\partial\left[d_1 F_X(d_1)\right]}{\partial d_1} + (1-b_v)\left[E[\tilde{X}]\frac{\partial F_X(d_1)}{\partial d_1} - \sigma_X^2\frac{\partial f_X(d_1)}{\partial d_1}\right] - b_f\frac{\partial F_X(d_1)}{\partial d_1} - \right. \\ &\left. -\lambda\ \text{Cov}(\tilde{X},\tilde{R}_m)\left[(1-b_v)\frac{\partial F_X(d_1)}{\partial d_1} + b_f\frac{\partial f_X(d_1)}{\partial d_1} + b_v\frac{\partial\left[d_1 f_X(d_1)\right]}{\partial d_1}\right]\right\} \qquad (1.36)\end{aligned}$$

After performing differentiations, substituting $\frac{\partial F_X(d_1)}{\partial d_1}$ by $f_X(d_1)$ and $\frac{\partial f_X(d_1)}{\partial d_1}$ by $-(\frac{d_1 - E[\tilde{X}]}{\sigma_X^2})f_X(d_1)$, and collecting terms, (1.36) becomes:

$$\begin{aligned} \frac{\partial D}{\partial d_1} &= \frac{1}{R_f}\left\{1 - F_X(d_1) - (b_f + b_v d_1)f_X(d_1) - \right. \\ &\left. -\lambda\ \text{Cov}(\tilde{X},\tilde{R}_m)f_X(d_1)\left[1 - (\frac{d_1 - E[\tilde{X}]}{\sigma_X^2})(b_f + b_v d_1)\right]\right\} \qquad (1.37)\end{aligned}$$

Setting (1.37) equal to zero gives:

$$\begin{aligned} 1 - F_X(d_1) &= (b_f + b_v d_1)f_X(d_1) + \\ &+\lambda\ \text{Cov}(\tilde{X},\tilde{R}_m)f_X(d_1)\left[1 - (\frac{d_1 - E[\tilde{X}]}{\sigma_X^2})(b_f + b_v d_1)\right] \qquad (1.38)\end{aligned}$$

The second derivative, $\partial^2 D/\partial d_1^2$, can be calculated by differentiating (1.37) with respect to d_1:

$$\begin{aligned} \frac{\partial^2 D}{\partial d_1^2} &= \frac{f_X(d_1)}{R_f}\left\{-(1+b_v) + \left(\frac{d_1 - E[\tilde{X}]}{\sigma_X^2}\right)\left[(b_f + b_v d_1) + \right.\right. \\ &\left.\left. +\lambda\ \text{Cov}(\tilde{X},\tilde{R}_m)\left(1 - (b_f + b_v d_1)\left(\frac{d_1 - E[\tilde{X}]}{\sigma_X^2} - b_v\right)\right)\right]\right\} \qquad (1.39)\end{aligned}$$

An inspection of (1.39) shows that $\partial^2 D/\partial d_1^2 < 0$ for any $d_1 < E[\tilde{X}]$. Hence, as long as the first condition is met inside the interval $0 \leq D < A$, d_1^c corresponds to a maximum.

According to Rolle's theorem, d_1^c exists only when the right-hand side (RHS) of (1.38) can assume values greater than the left-hand side (LHS). This is because at low values of d_1, (LHS) $>$ (RHS) and $\partial D/\partial d_1 > 0$. Thus, values of d_1 that satisfy (LHS) $<$ (RHS) inside the interval $0 \leq D < A$, (or within $0 \leq d_1 < A(1 + \textit{Int})$) assure that $\partial D/\partial d_1 = 0$ at some finite point d_1^c. Therefore, D^c can be calculated by substituting the promised amount d_1^c that satisfies (1.38) into (1.16).

Eq. 1.38 can be arranged to show how the level of bankruptcy costs affects the existence of the debt capacity of an owning company:

$$b_f + b_v d_1 \quad > \quad \frac{1 - F_X(d_1) - \lambda \text{ Cov}(\tilde{X}, \tilde{R}_m) f_X(d_1)}{f_X(d_1)\left[1 - (\frac{d_1 - E[\tilde{X}]}{\sigma_X^2})\lambda \text{ Cov}(\tilde{X}, \tilde{R}_m)\right]} \tag{1.40}$$

Thus, if the bankruptcy costs satisfy the above condition and $0 \leq d_1 < A(1 + \mathit{Int})$ then there is a finite limit on the owning company's debt capacity. If bankruptcy is costless, but there is still the possibility the company might go bankrupt, debt capacity exists as long as $\lambda \text{ Cov}(\tilde{X}, \tilde{R}_m) > \frac{1 - F_X(d_1)}{f_X(d_1)}$.

Notice that the numerator in (1.40), $1 - F_X(d_1) - \lambda \text{ Cov}(\tilde{X}, \tilde{R}_m) f_X(d_1)$, is the same as the certainty equivalent of one dollar associated with the occurrence of bankruptcy, $E[\delta_b] - \lambda \text{ Cov}(\delta_b, \tilde{R}_m)$ (see (A.11) and (A.20). Thus, in the extreme where bankruptcy becomes certain, the numerator in (1.40) becomes zero and from (1.38) we see that RHS $>$ LHS. Consequently, at the same extreme, $\partial D/\partial d_1$ is reduced to:

$$\frac{\partial D}{\partial d_1} \quad = \quad \frac{f_X(d_1)}{R_f}\left\{(1 - b_v)(d_1 - E[\tilde{X}]) - \lambda \text{ Cov}(\tilde{X}, \tilde{R}_m)\left(-b_f \frac{d_1 - E[\tilde{X}]}{\sigma_X^2} + b_v\right)\right\} \tag{1.41}$$

The above equation illustrates that, when bankruptcy is certain and costly, $\partial D/\partial d_1$ is always negative. Hence, **debt capacity is always reached (*i.e.*, $\partial D/\partial d_1 = 0$) before bankruptcy becomes certain. This means, that in the presence of bankruptcy costs that satisfy (1.40) within the interval $0 \leq d_1 < A(1+\mathit{Int})$, the owning company can never borrow 100% of the project's costs even if it wants to. Promising to pay more in the future (*i.e.*, increasing d_1 beyond d_1^c) does not increase D because of the higher risk of bankruptcy and its associated costs.**

1.5 Comparing The Above Formulation With The MM Model

Miller and Modigliani (MM), in their 1963 seminal paper, indicate that the market value of a levered company is a linearly increasing function of financial leverage with a slope that corresponds to the tax rate of the company and is equal to the value of the company if it were unlevered plus the present value of its tax credits, that is,

$$V' = V'_u + \text{PV}(T\mathit{Int}\$) \tag{1.42}$$

where V' is the market value of the company, V'_u is the value of the company if it were all-equity financed, T is the company's tax rate, and *Int*\$ is the interest in dollars paid on the debt. Notice that V' and V'_u are those variables that satisfy MM's equation. The notation V and V_u is reserved for those variables developed in this research.

According to the formulation developed in the previous section, the market value of a project can be calculated by summing the market values of its debt (Eq. 1.26) and equity (Eq. 1.35),

$$\begin{aligned} V &= \frac{1}{R_f}\Big\{ \Big[(1-T)E[\tilde{X}] + Td_1 + T(A-D)\Big](1-F_X(d_1)) + \\ &+(1-T)\sigma_X^2 f_X(d_1) + (1-b_v)\Big\{E[\tilde{X}](F_X(d_1)-F_X(b')) + \sigma_X^2(f_X(b')-f_X(d_1))\Big\} - \\ &-b_f(F_X(d_1)-F_X(b')) - \lambda\,\text{Cov}(\tilde{X},\tilde{R}_m)\Big[(1-T)(1-F_X(d_1)) + T(A-D)f_X(d_1) + \\ &+(1-b_v)(F_X(d_1)-F_X(b')) + (b_f+b_v d_1)f_X(d_1)\Big]\Big\} \end{aligned} \tag{1.43}$$

To determine the corresponding V_u, i.e. the market value of an unlevered project, it is necessary to set $D = 0$ as follows:

$$V_u = \frac{1}{R_f}\Big\{(1-T)E[\tilde{X}] + TA - (1-T)\lambda\,\text{Cov}(\tilde{X},\tilde{R}_m)\Big\} \tag{1.44}$$

Substituting (1.44) into (1.43) and rearranging terms provides an equation that relates V and V_u (as the MM formula does):

$$\begin{aligned} V &= V_u + \frac{1}{R_f}[T(d_1-D)(1-F_X(d_1)] + \frac{1}{R_f}\Big\{ \Big[(1-T)E[\tilde{X}] + TA\Big](-F_X(d_1)) + \\ &+(1-T)\sigma_X^2 f_X(d_1) + (1-b_v)\Big\{E[\tilde{X}](F_X(d_1)-F_X(b')) + \sigma_X^2(f_X(b')-f_X(d_1))\Big\} - \\ &-b_f(F_X(d_1)-F_X(b')) - \lambda\,\text{Cov}(\tilde{X},\tilde{R}_m)\Big[(1-T)(-F_X(d_1)) + T(A-D)f_X(d_1) + \\ &+(1-b_v)(F_X(d_1)-F_X(b')) + (b_f+b_v d_1)f_X(d_1)\Big]\Big\} \end{aligned} \tag{1.45}$$

For the case of taxes and no possibility of bankruptcy (i.e., $b_f = b_v = 0$, $F_X(d_1) = F_X(b') = 0$, which implies $f_X(d_1) = f_X(b') = 0$), the market value of the project given by (1.45) becomes

$$V = V_u + \frac{T(d_1-D)}{R_f} \tag{1.46}$$

which is equivalent to the market value of the project given by Miller and Modigliani (Eq. 1.42). However, in the case of taxes and costless (but possible) bankruptcy, (1.45) becomes

$$\begin{aligned} V &= V_u + \frac{T(d_1-D)}{R_f} + \frac{1}{R_f}\Big\{ \Big[(1-T)E[\tilde{X}] + T(d_1-D) + TA\Big](-F_X(d_1)) + \\ &+\sigma_X^2(f_X(b') - Tf_X(d_1)) - \lambda\,\text{Cov}(\tilde{X},\tilde{R}_m)\,[TF_X(d_1) - F_X(b') + \\ &+T(A-D)f_X(d_1)]\Big\} \end{aligned} \tag{1.47}$$

and does not provide the same results as the MM model. The additional terms only disappear if there is zero probability of bankruptcy. Therefore, the simple possibility of bankruptcy is sufficient to violate the MM model. Moreover, the fact that the sum of the additional terms is negative suggests that the maximization of debt financing might not necessarily lead to the maximum market value for a levered project as it does in the MM model. In fact, nobody should expect the two models (i.e., the MM model and the model developed here) to give the same answers as they are based on different assumptions. The MM model considers a perpetual multi-project company. The projects are financed on a recourse basis and the possibility of the company going bankrupt because one of its projects does is almost non-existent. The formulation developed here considers a single-period single-project company where the project is financed without collaterals to debtholders. Therefore, bankruptcy is much more likely and is explicitly considered.

In the presence of taxes and bankruptcy costs, the market value of a levered project, given by (1.45), can be seen as the value of the project as if it were unlevered plus the expected value of the tax subsidies received on the interest part of the debt minus a collection of terms that represents an ‘‘option’’ on bankruptcy costs. The analogy of the effect of bankruptcy costs on the valuation of a levered project with an ‘‘option’’ was first noted by Kim (1978) and can be illustrated by considering a group who receives the bankruptcy costs. This group holds a claim (‘‘option’’) on the project: if the project goes bankrupt (i.e., $\tilde{X} < d_1$) they exercise it and receive the bankruptcy costs; if the project does not go bankrupt they do not exercise it and receive nothing.

According to Kim (1978), the implication of the above statement is that the ownership interests on a levered project is not held by three distinct parties (i.e., equityholders, bondholders, and the government) as suggested by Miller and Modigliani, but rather by four groups: equityholders, bondholders, the government, and the holder of the option on bankruptcy costs. Therefore, the before-tax market value of a project, $V_{\tilde{X}} = (E[\tilde{X}] - \lambda \text{ Cov}(\tilde{X}, \tilde{R}_m))/R_f$, can be represented as

$$V_{\tilde{X}} = S + D + V_{gov} + V_{opt} \tag{1.48}$$

where V_{gov} is value of the government’s ‘‘ownership,’’ and V_{opt} is the value of the ‘‘option’’ on bankruptcy costs. The market value of the ‘‘option’’ on bankruptcy costs, V_{opt}, is given by:

$$V_{opt} = \frac{E[\tilde{B}] - \lambda \text{ Cov}(\tilde{B}, \tilde{R}_m)}{R_f} \tag{1.49}$$

where $\tilde{B}$ represents the uncertain cost of bankruptcy as defined in (1.13). Alternatively, $\tilde{B}$ can be expressed in the following equation form:

$$\tilde{B} = \delta_b \delta_q (b_f + b_v \tilde{X}) + (1 - \delta_b)\delta_q \tilde{X} \tag{1.50}$$

Solving the above equation with a similar procedure to that used to calculate the market values of debt and equity gives:

$$\begin{aligned} V_{opt} \;&=\; \frac{1}{R_f}\Big\{ b_f\,(F_X(d_1) - F_X(b')) + b_v \Big[E[\tilde{X}]\,(F_X(d_1) - F_X(b'))\; + \\ &\quad +\sigma_X^2\,(f_X(b') - f_X(d_1))\Big] + E[\tilde{X}]F_X(b') - \sigma_X^2 f_X(b') - \\ &\quad -\lambda \text{ Cov}(\tilde{X}, \tilde{R}_m)\Big[-f_X(d_1)(b_f + b_v d_1) + b_v F_X(d_1) + (1 - b_v)F_X(b')\Big]\Big\} \end{aligned} \tag{1.51}$$

From (1.48), the market value of the government’s ‘‘ownership’’ is

$$V_{gov} = V_{\tilde{X}} - S - D - V_{op} \tag{1.52}$$

Substituting (1.26), (1.35), and (1.51) into the above equation yields

$$\begin{aligned} V_{gov} \;&=\; \frac{T}{R_f}\Big\{\Big[E[\tilde{X}] - d_1 - (A - D)\Big]\,(1 - F_X(d_1)) + \sigma_X^2 f_X(d_1) - \\ &\quad -\lambda \text{ Cov}(\tilde{X}, \tilde{R}_m)\,[(1 - F_X(d_1)) - (A - D)f_X(d_1)]\Big\} \end{aligned} \tag{1.53}$$

From numerical analysis one can conclude that V_{opt} increases and V_{gov} decreases as the promised debt amount (d_1) increases. Therefore, the level of debt financing does matter, and there is such a thing as debt capacity and optimal capital structure.

1.6 Optimal Capital Structure

The optimal financial structure of an owning company is defined here as the combination of debt and equity that achieves a financial objective. Two such objectives are investigated here: maximizing the return on the equityholders’ investment (*ROE*) and maximizing the project’s net present value (*NPV*).

The *ROE* is calculated by dividing the end-of-period value of the project after all obligations have been paid (*i.e.*, expenses, debt and taxes) by the amount initially infused by project investors, that is:

$$\widetilde{ROE} = \frac{\tilde{S}_1}{A - D} \tag{1.54}$$

In order to determine the financial structure that maximizes the return to project investors it is necessary to follow a procedure similar to the one used to determine debt capacity, that is, to set $\partial E[\widetilde{ROE}]/\partial d_1 = 0$ and to verify that $\partial^2 E[\widetilde{ROE}]/\partial d_1^2 < 0$.

Differentiating $\dfrac{E[\tilde{S}_1]}{A-D}$ with respect to d_1 gives:

$$\frac{\partial E[\widetilde{ROE}]}{\partial d_1} = \frac{\partial}{\partial d_1}\left(\frac{E[\tilde{S}_1]}{(A-D)}\right) = \frac{\frac{\partial E[\tilde{S}_1]}{\partial d_1}(A-D) + E[\tilde{S}_1]\frac{\partial D}{\partial d_1}}{(A-D)^2} \tag{1.55}$$

$$\begin{aligned} &= \frac{1}{(A-D)^2}\Bigg\{(A-D)\Bigg[-(1-T)(1-F_X(d_1)) - \\ &\quad -T\left((A-D)f_X(d_1) + \frac{\partial D}{\partial d_1}(1-F_X(d_1))\right)\Bigg] + E[\tilde{S}_1]\frac{\partial D}{\partial d_1}\Bigg\} \end{aligned} \tag{1.56}$$

As the optimal capital structure occurs when $\dfrac{\partial E[\widetilde{ROE}]}{\partial d_1} = 0$, (1.56) yields:

$$\begin{aligned} E[\tilde{S}_1]\frac{\partial D}{\partial d_1} &= (A-D)\Bigg\{(1-T)(1-F_X(d_1)) + \\ &\quad +T\left[(A-D)f_X(d_1) + \frac{\partial D}{\partial d_1}(1-F_X(d_1))\right]\Bigg\} \end{aligned} \tag{1.57}$$

and solving for $\dfrac{\partial D}{\partial d_1}$ gives:

$$\frac{\partial D}{\partial d_1} = \frac{(1-T)(1-F_X(d_1)) + T(A-D)f_X(d_1)}{\frac{E[\tilde{S}_1]}{A-D} - T(1-F_X(d_1))} \tag{1.58}$$

Note that the numerator and the denominator of (1.58) are positive for $d_1 < d_1^c$ and $A > D$. (In order to prove that the denominator of (1.58) is always positive, it is only necessary to show that $E[\tilde{S}_1] > T(A-D)(1-F_X(d_1))$ which is trivial because $0 < T(1-F_X(d_1)) < 1$ and $E[\tilde{S}_1] \geq A - D$ for a positive *NPV* project.) Consequently, when $\partial E[\widetilde{ROE}]/\partial d_1 = 0$ we always have $\partial D/\partial d_1 > 0$. Thus, the company's optimal capital structure always occurs before its debt capacity is reached, $d_1^{ROE} < d_1^c$, where d_1^{ROE}, the promised debt amount that maximizes the return to equityholders, is the value of d_1 that satisfies (1.58). Therefore, **when debt capacity**

does not allow 100% debt financing ($A > D$), an owning company that wants to maximize its return on investment should borrow at less than debt capacity. If the project's debt capacity allows 100% debt financing (*i.e.*, $D = A$), (1.58) gives $\partial D/\partial d_1 = 0$ and the optimal capital structure occurs at 100% debt financing.

A similar analysis can be undertaken for the objective of maximizing the project's net present value. From (1.27) we see that the optimal capital structure that maximizes *NPV* is exactly the same as the amount of debt and equity that maximizes the wealth of the equityholder in traditional finance (Brealey and Myers 1991), that is,

$$\frac{\partial NPV}{\partial d_1} = \frac{\partial V}{\partial d_1} = \frac{\partial D}{\partial d_1} + \frac{\partial S}{\partial d_1} = 0 \tag{1.59}$$

The objective of maximizing the equityholders' wealth does not usually provide the same ''optimal'' capital structure as the objective of maximizing their returns. The two objectives provide similar results only when

$$\left.\frac{\partial V}{\partial d_1}\right|_{d_1^{ROE}} = \left.\frac{\partial E[\widetilde{ROE}]}{\partial d_1}\right|_{d_1^{ROE}} = 0 \tag{1.60}$$

and this implies:

$$E[\widetilde{ROE} \mid d_1 = d_1^{ROE}] \approx R_f \tag{1.61}$$

In order to see this, substitute $\dfrac{\partial D}{\partial d_1}$ by $-\dfrac{\partial S}{\partial d_1}$ (from (1.59)), $E[\tilde{S}_1] = E[\widetilde{ROE}](A - D)$ (from (1.54)) into (1.55) and let d_1^V be the value of d_1 that satisfies (1.59),

$$\left.\frac{\partial E[\widetilde{ROE}]}{\partial d_1}\right|_{d_1^V} = \frac{1}{A-D}\left.\left[\frac{\partial E[\tilde{S}_1]}{\partial d_1} - E[\widetilde{ROE}]\frac{\partial S}{\partial d_1}\right]\right|_{d_1^V} \tag{1.62}$$

Setting the right-hand side of (1.62) equal to zero gives:

$$E[\widetilde{ROE}] = \frac{\partial E[\tilde{S}_1]/\partial d_1}{\partial S/\partial d_1} \tag{1.63}$$

Substituting $\dfrac{\partial E[\tilde{S}_1]}{\partial d_1}$ by the definition given in (1.28) yields:

$$E[\widetilde{ROE}] = Rf + \lambda \frac{\frac{\partial}{\partial d_1}\ \text{Cov}(\tilde{S}_1, \tilde{R}_m)}{\frac{\partial S}{\partial d_1}} \approx R_f \tag{1.64}$$

The above derivation implies that if $d_1^V \neq d_1^{ROE}$ then the objective of maximizing the $E[\widetilde{ROE}]$ does not provide an ''optimal capital structure'' similar to the objective of maximizing stock-

Project		Market	
Variable	Value	Variable	Value
(1)	(2)	(3)	(4)
A	\$ 2,170	$E[\tilde{R}_m]$	1.14
$E[\tilde{X}]$	\$ 2,750	σ_m	0.25
σ_X	\$ 800	T	0.35
ρ_{X,R_m}	0.70	R_f	1.06
b_f	\$ 100		
b_v	0.30		

Table 1-1: Input Parameters for the Example Project (all $ values are in millions)

holders' wealth. More specifically, $d_1^V < d_1^{ROE}$ if:

$$E[\widetilde{ROE}]\Big|_{d_1^{ROE}} > Rf + \lambda \frac{\frac{\partial}{\partial d_1}\ \text{Cov}(\tilde{S}_1, \tilde{R}_m)}{\frac{\partial S}{\partial d_1}} \approx R_f \tag{1.65}$$

In other words, since (1.65) should always be true, **the maximization of return on equity investment always allows more borrowing than the maximization of the company's net present value.** This is made evident by the following example.

1.7 Example

This section presents an example to illustrate the concepts developed in previous sections. Table 1-1 shows the input parameters necessary for the determination of the debt capacity and the optimal capital structure of a privately-financed project and displays the specific values assumed for the parameters in this example.

Table 1-2 contains the numerical values of D, S, V, *NPV*, D/A, $E[\tilde{r}_D]$, *Int*, $E[\tilde{r}_S]$, $E[\widetilde{Roe}]$, $\partial D/\partial d_1$, $\partial E[\widetilde{ROE}]/\partial d_1$, and $\partial V/\partial d_1$ for different d_1 values. The present (market) values of debt and equity, D and S, are calculated from (1.26) and (1.35). The present value of the project is $V = D + S$, and $NPV = V - A$. Of course, these are only valid for $S \geq 0$. The percentage of debt financing used in the project, D/A, is the ratio between the present value of the project's debt (*i.e.*, the amount of money debtholders will provide to the project) and the initial cost of the project.

The effective return on debt, $E[\tilde{r}_D]$, is the expected return for the debtholders. It can be determined by substituting (1.25) into (1.1). Thus, $D(1 + E[\tilde{r}_D])$ is the repayment amount debtholders expect to receive at the end of the period. The promised return on debt, *Int*, is the interest rate debtholders would charge the owning company in order to lend them D. *Int* is calculated as the ratio between d_1 and D, minus one.

The required return on equity, $E[\tilde{r}_S]$, is the return investors would expect to receive if they had invested in an openly-traded asset that presents the same degree of risk as the privately-financed project (*i.e.*, $\beta_{\text{asset}} = \beta_{\text{project}}$). The expected return on equity investment, $E[\widetilde{Roe}]$, is the ratio between the expected end-of-period value of the project after all obligations have been paid and the amount infused by investors at the beginning of the period, minus one. The rates of change, $\partial D/\partial d_1$, $\partial E[\widetilde{ROE}]/\partial d_1$, and $\partial V/\partial d_1$, are calculated from (1.37), (1.58), and (1.59) respectively.

Figure 1-3 shows D, S, and V, as d_1 increases. According to (1.16), the value of the debt is the amount of money debtholders expect to receive at the end of the period minus the systematic risk premium on the project's debt (*i.e.*, the amount lenders charge to buy part of the project's systematic operating risk premium from the owning company), divided by R_f. As long as $d_1 < d_1^c$, any increment on the promised debt amount, increases the amount debtholders expect to receive at the end of the period more than it increases the amount they charge to take the risk from the owning company. Thus, any increment in d_1 would increase both the nominal interest rate *Int* and the loan amount D; that is, in the interval $(0, d_1^c)$, the systematic risk premium on the project's debt would never dominate the expected debt repayment amount.

At $d_1 = d_1^c$, the market value of the debtholders' holdings reaches a maximum, therefore D^c is the maximum amount of money the owning company can borrow from debtholders (*i.e.*, the debt capacity of the project). At this point, a small increase in d_1 is completely offset by an appropriate increase in *Int* leaving D^c constant. If $d_1 > d_1^c$, debtholders would decrease the amount they would lend to the owning company because the amount charged by them to buy the risk from the company would always dominate the amount they expect to receive from debt

Promised debt amount d_1 (1)	Market value of debt D (2)	Market value of equity S (3)	Market value of project V (4)	Net Present Value *NPV* (5)	Debt financing D/A (6)	Effective return on debt $E[\tilde{r}_D]$ (7)	Promised return on debt *Int* (8)	Required return on equity $E[\tilde{r}_S]$ (9)	Return on equity investment $E[\widetilde{Roe}]$ (10)	$\partial D/\partial d_1$ (11)	$\partial E[\widetilde{ROE}]/\partial d_1$ (12)	$\partial V/\partial d_1$ (13)
0	0	2,293	2,293	123	0.000	0.060	0.060	0.111	0.174	0.94	5.79E-05	0.02
174	164	2,132	2,295	125	0.075	0.061	0.061	0.115	0.184	0.94	6.70E-05	0.02
347	327	1,971	2,298	128	0.151	0.061	0.062	0.119	0.197	0.94	7.79E-05	0.01
521	490	1,810	2,300	130	0.226	0.062	0.064	0.125	0.212	0.94	9.06E-05	0.01
694	651	1,649	2,301	131	0.300	0.062	0.066	0.131	0.229	0.93	1.05E-04	0.00
714	670	1,631	2,301	131	0.309	0.063	0.066	0.132	0.231	0.93	1.07E-04	0.00
868	811	1,489	2,300	130	0.374	0.064	0.070	0.139	0.248	0.91	1.21E-04	-0.01
1,042	968	1,329	2,297	127	0.446	0.066	0.076	0.149	0.271	0.89	1.36E-04	-0.02
1,215	1,119	1,172	2,291	121	0.516	0.069	0.086	0.161	0.295	0.85	1.47E-04	-0.04
1,389	1,264	1,018	2,282	112	0.582	0.073	0.099	0.176	0.321	0.80	1.47E-04	-0.07
1,562	1,397	871	2,268	98	0.644	0.078	0.118	0.194	0.345	0.73	1.24E-04	-0.10
1,736	1,517	732	2,249	79	0.699	0.085	0.145	0.215	0.362	0.64	5.65E-05	-0.13
1,800	1,556	684	2,240	70	0.717	0.088	0.157	0.224	0.364	0.60	1.49E-05	-0.14
1,910	1,619	605	2,223	53	0.746	0.093	0.180	0.241	0.360	0.53	-8.34E-05	-0.17
2,083	1,700	491	2,191	21	0.783	0.102	0.225	0.270	0.327	0.41	-3.19E-04	-0.21
2,179	1,736	434	2,170	0	0.800	0.107	0.255	0.289	0.289	0.34	-4.87E-04	-0.23
2,257	1,759	392	2,151	-19	0.811	0.112	0.283	0.305	0.245	0.28	-6.37E-04	-0.25
2,430	1,796	308	2,104	-66	0.828	0.122	0.353	0.343	0.106	0.15	-9.54E-04	-0.29
2,604	1,812	238	2,050	-120	0.835	0.132	0.437	0.386	-0.079	0.03	-1.14E-03	-0.33
2,659	1,812	219	2,031	-139	0.835	0.135	0.467	0.401	-0.143	0.00	-1.16E-03	-0.34
2,778	1,809	181	1,990	-180	0.834	0.142	0.536	0.435	-0.280	-0.06	-1.14E-03	-0.36
2,951	1,792	135	1,927	-243	0.826	0.150	0.647	0.489	-0.467	-0.13	-9.91E-04	-0.36
3,125	1,766	99	1,865	-305	0.814	0.155	0.769	0.552	-0.621	-0.17	-7.80E-04	-0.35
3,298	1,736	70	1,806	-364	0.800	0.159	0.900	0.624	-0.738	-0.18	-5.79E-04	-0.32
3,472	1,706	48	1,754	-416	0.786	0.160	1.035	0.708	-0.824	-0.17	-4.14E-04	-0.32

Table 1-2: Example Results (all $ in millions)

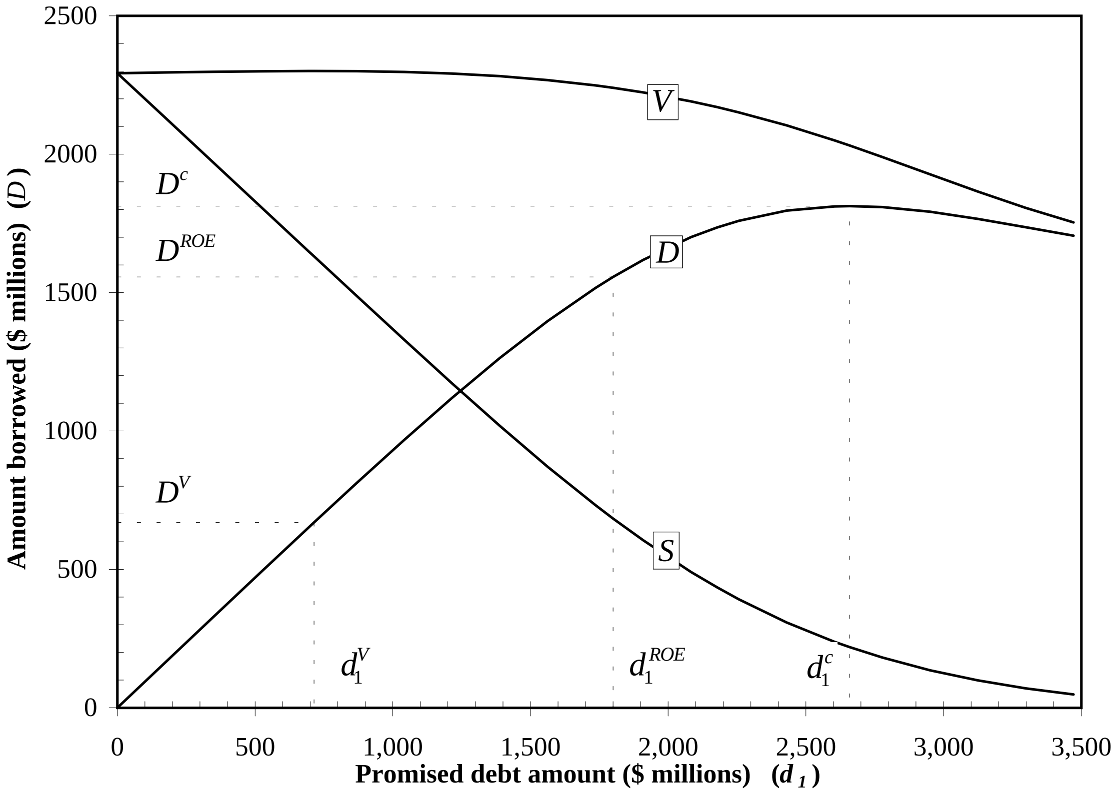

Figure 1-3: Present Values of V, D, and S as Functions of d_1

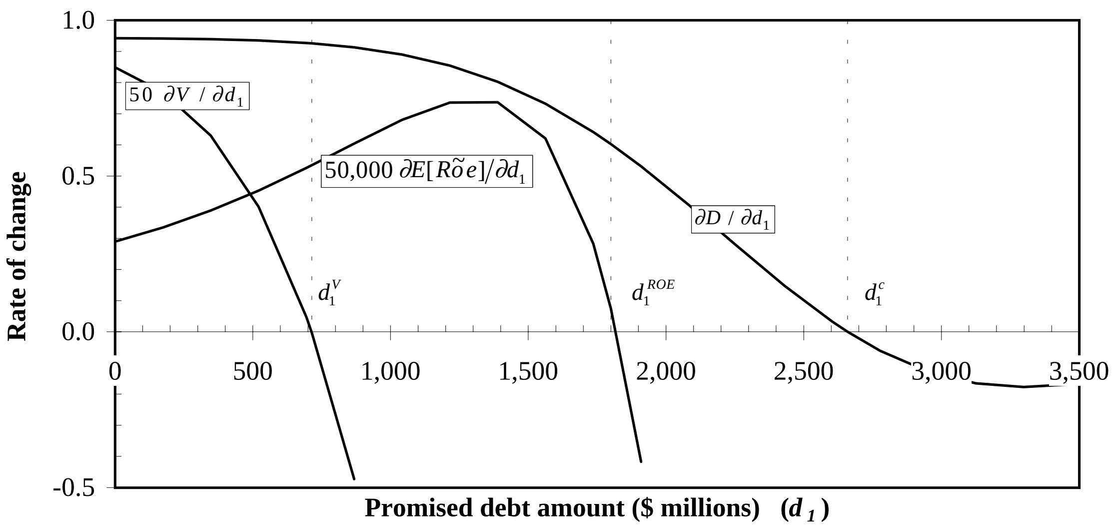

Figure 1-4: Rates of Change of D, $E[\widetilde{Roe}]$, and V as Functions of d_1

repayment; that is, an increase in d_1 results in such a large increase in *Int* that the value of the debt D actually decreases.

As d_1 increases, the value of the equity, S, decreases because: (i) the probability of bankruptcy increases and (ii) the amount infused by investors decreases. As the probability of bankruptcy increases, the likelihood that a project generates enough income to distribute earnings to investors, after paying for all financial obligations, decreases. If $d_1 > d_1^c$, investors would, theoretically, infuse more equity to finance the project (since D now decreases). However, the probability of bankruptcy more than offsets this increase in equity infusion and S would still decrease as d_1 increases past d_1^c.

Figure 1-3 shows that the value of the owning company, V, first increases slightly as d_1 increases, reaches a maximum at d_1^V and then decreases. Figure 1-4 shows the point d_1^V where $\partial V/\partial d_1 = \partial NPV/\partial d_1 = 0$ and thus, illustrates the existence a debt financing amount, D^V, that maximizes the value of the owning company as well as the *NPV* of the project.

The dashed lines in Figures 1-5 and 1-6 correspond to the expected values of the returns on the debt and equity of an openly-traded asset that has the same risk class as the privately-financed project. The solid lines represent the investors' expected rate of return on equity, the interest rate charged by debtholders, and the project's *NPV*. Figure 1-5 shows that the difference between *Int* and $E[\tilde{r}_D]$ widens as d_1 increases, and thus, illustrates how the premium charged by lenders to take some of the net income risks from the investors increases as d_1 increases. It also shows that the promised debt amounts d_1^V and d_1^{ROE} do indeed maximize the project's NPV and the investors' expected return on equity. Figure 1-3 shows the associated optimal debt amounts D^V and D^{ROE} (depending on which objective one chooses to maximize).

Moreover, Figures 1-5 and 1-6 show that $E[\widetilde{Roe}]$ can be larger, equal or smaller than $E[\tilde{r}_S]$. At point ''Z,'' $E[\widetilde{Roe}] = E[\tilde{r}_S]$ and the project has $NPV = 0$. Promised debt amounts smaller than the one corresponding to ''Z'' yield $E[\widetilde{Roe}] > E[\tilde{r}_S]$; consequently, the project has a positive *NPV* and investors earn more than the return required by the market (as they should because they are the ones that create value by making the project a reality).

Figure 1-6 is similar to Figure 1-5 but uses ''percent debt financing'' (*i.e.*, D/A) as the x-axis instead of the promised debt amount d_1. Thus, it illustrates the existence of a *''debt-capacity frontier''* at d_1^c, that limits the borrowing of the owning company. The reason, of course, is the

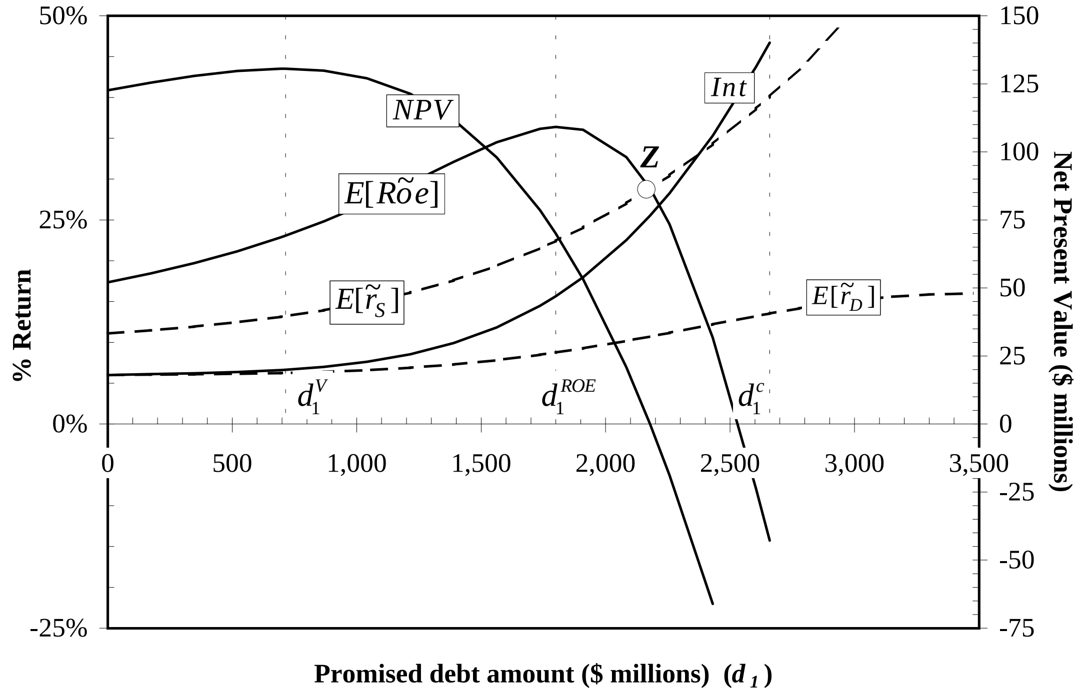

Figure 1-5: $E[\tilde{r}_D]$, *Int*, $E[\tilde{r}_S]$, $E[\widetilde{Roe}]$, and *NPV* as Functions of d_1

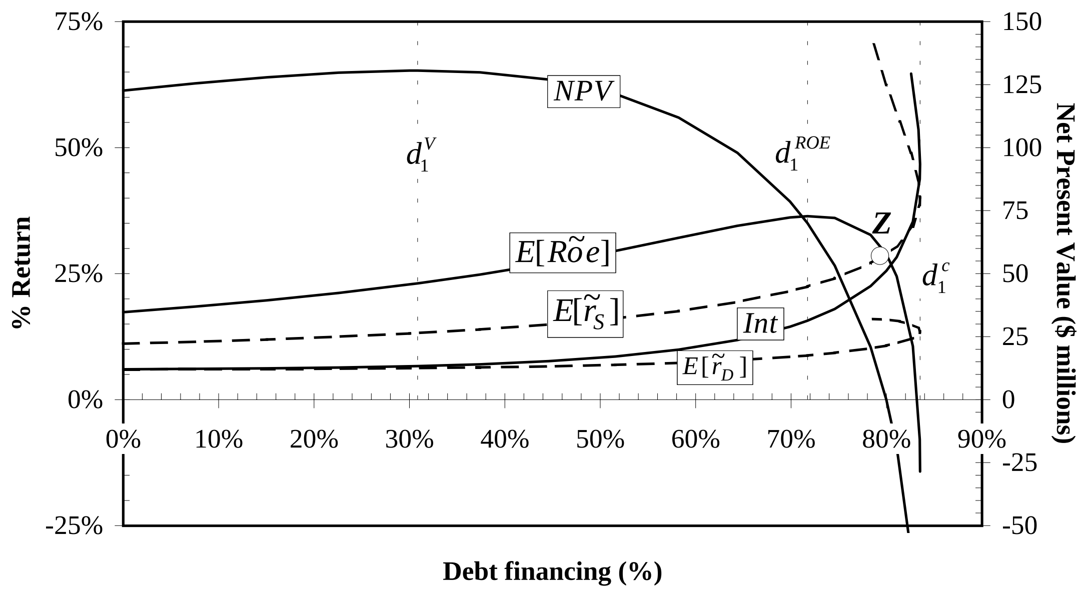

Figure 1-6: $E[\tilde{r}_D]$, *Int*, $E[\tilde{r}_S]$, $E[\widetilde{Roe}]$, and *NPV* as Functions of Percentage of Debt Financing

existence of a maximum debt D^c as shown in Figure 1-3. For example, if 78.3% of the project is financed through debt, then lenders would provide $1,700 M and would require a promised debt amount of either $2,083 M (if $d_1 < d_1^c$) or $3,510 M (if $d_1 > d_1^c$). Thus, depending on whether d_1 is smaller or larger than d_1^c, lenders would charge an interest of either 27% or 106% and would have an expect return of either 10.2% or 16.0%. It is obvious that, from the owning company's viewpoint, it would always prefer to promise to pay less for the money it borrows. Therefore, the company will never borrow more than the debt capacity of the project as it would always put it in a worse situation than if it borrows up to capacity. Similarly, the lenders would never be able to expect a return on the debt greater than the $E[\tilde{r}_D]$ that occurs at d_1^c.

Figure 1-6 also shows that both the project's *NPV*and $E[\widetilde{Roe}]$ decline rapidly as D approaches D^c. Thus, both objectives can be quite sensitive to the amount of debt used to finance the project and the owning company should take great care to avoid excessive borrowing even if lenders are willing to provide the debt.

1.8 Summary

Private promotion of projects is an alternative arrangement for developing and implementing many types of projects that range from civil infrastructure works to industrial facilities. The involvement of the private sector can provide two major benefits: (a) gains in efficiency arising from the business expertise offered by the private sector (*e.g.*, innovation, marketing and management skills) and greater incentives for the control of construction, operating, and maintenance costs; and (b) the provision of additional finance for economically justifiable projects. A more detailed discussion of the many benefits of privately-financed projects appears in (Dias, 1994).

The creation of a single-project company allows off-balance-sheet financing, which is advantageous to corporate equityholders because of their limited liability. This situation creates risky and ''expensive'' debt because debtholders increase their risk premiums to account for the probability that the owning company defaults. We have shown that there is a limit to what owning companies can borrow to finance a project (debt capacity). If bankruptcy is costly, this limit is reached before bankruptcy becomes certain. We have also shown that if equityholders want to maximize the project's net present value, or the return on their investment, then they should borrow less than the available debt capacity. We have also compared the objective of maximizing equity returns with the more traditional objective of maximizing equityholders' wealth (*i.e.*, net present value). From this it was shown that the capital structure used to maximize returns would allow more debt financing than the more traditional objective of maximizing wealth and *NPV*. Most importantly, we have illustrated that the project's *NPV* and the equityholders return can be quite sensitive to the selected debt-equity ratio and decline rapidly as the owning company borrows more than the optimal amount in an attempt to reach the project's debt capacity level D^c. Thus, the issue of optimal capital structure merits significant attention at the project evaluation phase.

A one-period project was assumed in order to keep the mathematical analysis as simple as possible. The same general conclusions, however, are also valid for multi-period projects. The analysis, in this case, is best undertaken using numerical methods (a simple spreadsheet model may be sufficient) that apply the basic mathematical results presented above over many periods.

Chapter 2

The Effect of Guarantees on the Debt Capacity and on the Capital Structure of Privately-Financed Infrastructure Projects

''Life is the summation of confusions. The more confused you are, the more alive you are. When you are not confused any longer, you are dead.''

Turan Gonen

Guarantees are formal assurances provided by the host government with the objective of reducing and limiting the project risks faced by the participants of a privately-promoted infrastructure project. The government, by furnishing a guarantee, expresses a commitment towards the project and assumes responsibility to fulfill its conditions. Examples of guarantees include:

- Convertibility guarantee --- mechanism that assures foreign investors that they will be authorized to convert local currency earnings into foreign currency;

- Foreign exchange guarantee --- government assurance that the owning company will be indemnified for project revenue losses due to currency rate fluctuations;

- ‘‘No-second-facility’’ guarantee --- government assurance that it will not construct or will not allow others to construct and to explore competing facilities;

- Revenue guarantee --- covenant that guarantees a minimum operating income or a minimum demand volume to the owning company.

In this chapter we develop all the formulation necessary to calculate the market value of a project (and also of its debt and equity components) in the presence of two kinds of revenue guarantees: production and minimum revenue. In a production guarantee, the government guarantees a minimum demand for the service provided by the facility. For example, in the BOT power-plant project developed in Shajiao, China, the government agreed to purchase a minimum quantity of electricity over the 10-year concession period. On a minimum-revenue guarantee, the government assures a certain level of revenue and hence, in case the facility is not able to reach this level, provides the necessary revenues. In the case of the Malaysian North-South Expressway, the government agreed to provide a loan facility if the traffic volume fell below an assured minimum level in the first 17 years of the concession period.

The same marketing equilibrium approach used in the last chapter is applied here, the only difference is that the formulation is derived based on project revenues instead of its net operating income. This is because some forms of guarantees secure levels of revenues instead of net operating income and thus, require a formulation based on revenues. Next, we show that the presence of guarantees increases the debt capacity of the owning company as the project becomes less risky. Finally, we show that the project’s net present value and the investors’ return on investment also increase due to the guarantees. An example illustrates these concepts.

2.1 Valuation Based on Project Revenues

In the last chapter, all the formulation about debt capacity and optimal capital structure was derived based on the project’s net operating income, $\tilde{X}$. From hereon, all the derivation will be based on the variable ‘‘project revenues’’ instead of the net operating income. This is because some forms of guarantees (e.g., minimum-revenue) secure levels of revenue instead of net operating income and hence, require a formulation based on a more ‘‘basic’’ variable than the net operating income. Nonetheless, project revenues by itself is not sufficient to develop all the formulation necessary to determine the project's debt capacity and optimal capital structure as it does not provide the essential information of how much money there is at the end of the period to repay the debt and to be distributed to investors. The other variable that needs to be considered is ''project expenses.'' In this study, expenses are assumed to be a linear function of revenues, that is:

$$\widetilde{Exp} = e_f + e_v \widetilde{Rev} \tag{2.1}$$

where $\widetilde{Exp}$ represents the uncertain value of expenses at the end of the period, e_f is the fixed amount of expenses expressed in monetary units, e_v is the variable component expressed as a fraction of the project revenues $\widetilde{Rev}$. Thus, the net operating income can be redefined as:

$$\tilde{X} = (1 - e_v)\widetilde{Rev} - e_f \tag{2.2}$$

In order to include the possibility of having either a production or a minimum-revenue guarantee in the formulation derived in the previous chapter, it is necessary to define the following variables:

- $\widetilde{Rev}_g$ is the maximum value between the project revenues and the project guarantee, that is:

$$\widetilde{Rev}_g = \begin{cases} \widetilde{Rev} & \text{if } \widetilde{Rev} \geq g \\ g & \text{if } \widetilde{Rev} < g \end{cases} \tag{2.3}$$

- b'' is the maximum amount of $\widetilde{Rev}_g$ that would not provide any return to debtholders (*i.e.*, all the money generated by the project would be used to pay bankruptcy costs), and is calculated as:

$$b'' = \frac{b_f + e_f(1 - b_v)}{(1 - b_v)(1 - e_v)} \tag{2.4}$$

- a'' is the minimum amount of $\widetilde{Rev}_g$ that does not cause the project to face bankruptcy (*i.e.*, the money generated by the project would be sufficient to pay project expenses plus debt expenses). Mathematically,

$$a'' = \frac{d_1 + e_f}{1 - e_v} \tag{2.5}$$

In the next section, the actual market values of debt and equity for a privately-financed infrastructure project secured by a production guarantee will be examined. Such assessment will be performed for three different levels of guarantees:

- guarantee $< b''$;

- $b'' \leq$ guarantee $< a''$; and

- guarantee $\geq a''$

The following section will perform a similar assessment on minimum-revenue guarantees. Note the formulation derived in this chapter requires that $a'' \geq b''$, or equivalently, $d_1 \geq b_f/(1-b_v)$.

2.2 Production Guarantee

A production guarantee assures a minimum demand for the service provided by a privately promoted project. Therefore, the net operating income for a project that has such a guarantee is:

$$\tilde{X}_p = \begin{cases} (1-e_v)\widetilde{Rev} - e_f & \text{if } \widetilde{Rev} \geq g \\ (1-e_v)g - e_f & \text{if } \widetilde{Rev} < g \end{cases} \tag{2.6}$$

Alternatively, $\tilde{X}_p$ can be expressed in the following equation form:

$$\tilde{X}_p = [(1-e_v)\widetilde{Rev} - e_f]\delta_k + [(1-e_v)g - e_f](1-\delta_k) \tag{2.7}$$

$$= (1-e_v)[\delta_k\widetilde{Rev} + g(1-\delta_k)] - e_f \tag{2.8}$$

where δ_k is a binary variable defined as follows:

$$\delta_k = \begin{cases} 0 & \text{if } \widetilde{Rev} < g \\ 1 & \text{if } \widetilde{Rev} \geq g \end{cases} \tag{2.9}$$

2.2.1 The Present Value of Debt

According to the CAPM, the loan amount D_p (*i.e.,* the present value of a project's debt as determined by the market) can be computed as:

$$D_p = \frac{E[\tilde{D}_{1,p}] - \lambda \operatorname{Cov}(\tilde{D}_{1,p}, \tilde{R}_m)}{R_f} \tag{2.10}$$

where $E[\tilde{D}_{1,p}]$ is the expected value of the debt at time 1 when the project is backed by a production guarantee; λ is the market price per unit of risk; $\operatorname{Cov}(\tilde{D}_{1,p}, \tilde{R}_m)$ is the covariance between the value of debt at time 1 and one-plus-the-rate-of-return-on-the-market ; and R_f is one-plus-the-risk-free-rate.

The end-of-period value of debt, $\tilde{D}_{1,p}$, depends on the end-of-period project net operating income, $\tilde{X}_p$, and can be expressed as:

$$\tilde{D}_{1,p} = \begin{cases} d_1 & \text{if } \tilde{X}_p \geq d_1 \\ \tilde{X}_p - \tilde{B} & \text{if } \tilde{B} \leq \tilde{X}_p < d_1 \\ 0 & \text{if } \tilde{X}_p < \tilde{B} \end{cases} \tag{2.11}$$

or equivalently,

$$\tilde{D}_{1,p} = \begin{cases} d_1 & \text{if } \widetilde{Rev}_g \geq a'' \\ (1-b_v)[(1-e_v)\widetilde{Rev}_g - e_f] - b_f & \text{if } b'' \leq \widetilde{Rev}_g < a'' \\ 0 & \text{if } \widetilde{Rev}_g < b'' \end{cases} \tag{2.12}$$

Thus, if the net cash flow at the end of the period is greater than the promised amount d_1, the debtholders receive the full debt payments. Otherwise, they receive the net cash flow minus the bankruptcy costs, provided this difference is positive, and nothing if the difference is negative (the entire net operating income is consumed by bankruptcy costs). Figure 2-1 shows the value of $\tilde{D}_1$ as a function of $\tilde{X}$ and $\widetilde{Rev}$ for $b'' \leq$ production guarantee $< a''$. The graph on the left, shows $\tilde{D}_1$ (when bankruptcy costs are not considered) and $\tilde{B}$, without a guarantee. The graph on the right, illustrates the actual values of $\tilde{D}_1$ given by (2.12), when a production guarantee and bankruptcy costs are considered.

Alternatively, $\tilde{D}_{1,p}$ can be expressed in the following equation form:

$$\tilde{D}_{1,p} = d_1(1-\delta_b)\delta_q + \delta_b\delta_q\tilde{X}_p - \delta_b\delta_q(b_f + b_v\tilde{X}_p) \tag{2.13}$$

where δ_b and δ_q are binary variables defined as follows:

$$\delta_b = \begin{cases} 0 & \text{if } \widetilde{Rev}_g \geq a'' = \frac{d_1+e_f}{1-e_v} \\ 1 & \text{if } \widetilde{Rev}_g < a'' \end{cases} \tag{2.14}$$

$$\delta_q = \begin{cases} 0 & \text{if } \widetilde{Rev}_g < b'' = \frac{b_f+e_f(1-b_v)}{(1-b_v)(1-e_v)} \\ 1 & \text{if } \widetilde{Rev}_g \geq b'' \end{cases} \tag{2.15}$$

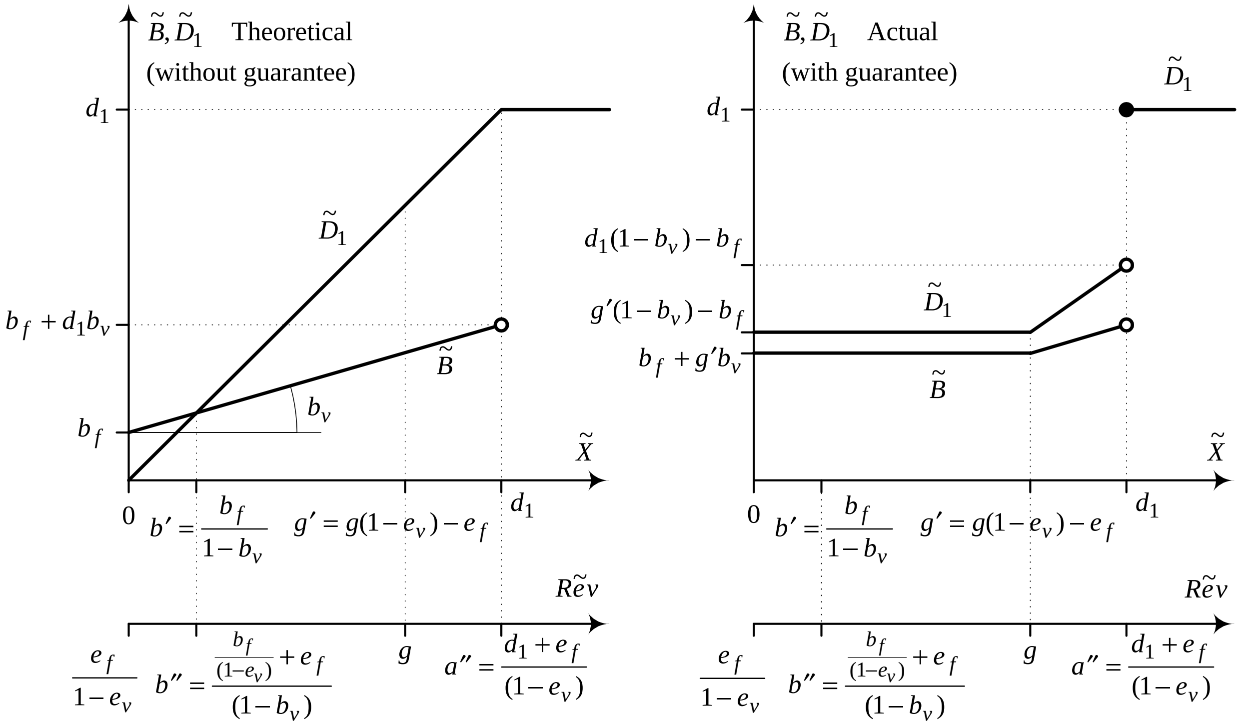

Figure 2-1: Value of debt ($\tilde{D}_1$) and bankruptcy costs ($\tilde{B}$) as a function of $\tilde{X}$ and $\widetilde{Rev}$

Substituting (2.8) into (2.13) gives:

$$\begin{aligned}\tilde{D}_{1,p} &= d_1(1-\delta_b)\delta_q + \delta_b\delta_q\left((1-e_v)[\delta_k\widetilde{Rev} + g(1-\delta_k)] - e_f\right) - \\ &\quad -\delta_b\delta_q\left[b_f + b_v\left((1-e_v)[\delta_k\widetilde{Rev} + g(1-\delta_k)] - e_f\right)\right] \qquad (2.16)\\ &= d_1(1-\delta_b)\delta_q + (1-e_v)(1-b_v)[\delta_b\delta_k\delta_q\widetilde{Rev} + g(\delta_b\delta_q - \delta_b\delta_k\delta_q)] - \\ &\quad -[e_f(1-b_v) + b_f]\delta_b\delta_q \qquad (2.17)\end{aligned}$$

$0 \leq$ **guarantee** $< b''$

Figure 2-2 shows the values of the different combinations of δ_b, δ_k and δ_q when the production guarantee is smaller than b''. Substituting these values in (2.17) gives:

$$\tilde{D}_{1,p} = d_1(1-\delta_b) + (1-e_v)(1-b_v)\delta_b\delta_q\widetilde{Rev} - [e_f(1-b_v) + b_f]\delta_b\delta_q \qquad (2.18)$$

The expected value of the end-of-period debt, $E[\tilde{D}_{1,p}]$, can be calculated in (2.18) as:

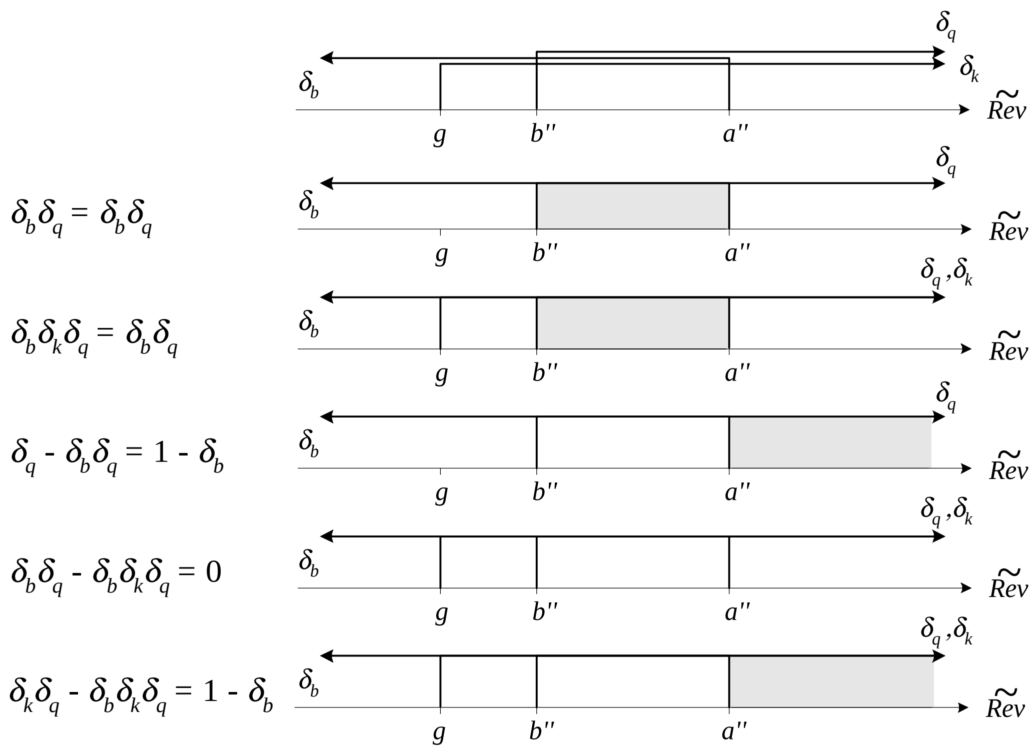

Figure 2-2: Combinations of δ_b, δ_k and δ_q When $0 \leq$ guarantee $< b''$

$$
\begin{aligned}
E[\tilde{D}_{1,p}] &= d_1(1 - E[\delta_b]) + (1 - e_v)(1 - b_v)E[\delta_b\delta_q\widetilde{Rev}] - \\
&\quad -[e_f(1 - b_v) + b_f]E[\delta_b\delta_q] \qquad (2.19)
\end{aligned}
$$

The expected values on the right-hand side of (2.19) are derived in Appendix A under the assumption that $\widetilde{Rev}$ follows a Normal distribution. Substituting (A.11), (A.13), and (A.14) into (2.19) and rearranging terms gives:

$$
\begin{aligned}
E[\tilde{D}_{1,p}] &= d_1\left(1 - F_{Rev}(a'')\right) + (1 - e_v)(1 - b_v)\Big\{E[\widetilde{Rev}]\left(F_{Rev}(a'') - F_{Rev}(b'')\right) + \\
&\quad +\sigma^2_{Rev}\left(f_{Rev}(b'') - f_{Rev}(a'')\right)\Big\} - \\
&\quad -[e_f(1 - b_v) + b_f]\left(F_{Rev}(a'') - F_{Rev}(b'')\right) \qquad (2.20)
\end{aligned}
$$

Therefore, the expected end-of-period payment to debtholders after bankruptcy costs is the full promised amount d_1 multiplied by the probability that the project does not go bankrupt plus the conditional expected value of the project operating income given that the project is bankrupt, $\{(1 - e_v)E[\widetilde{Rev}] - e_f\}\{F_{Rev}(a'') - F_{Rev}(b'')\} + (1 - e_v)\sigma^2_{Rev}\left(f_{Rev}(b'') - f_{Rev}(a'')\right)$, minus the expected value of the bankruptcy costs, $b_v(1 - e_v)\{E[\widetilde{Rev}]\left(F_{Rev}(a'') - F_{Rev}(b'')\right) +$

$\sigma^2_{Rev}\left(f_{Rev}(b'') - f_{Rev}(a'')\right)\} + [b_f - b_v e_f]\{F_{Rev}(a'') - F_{Rev}(b'')\}.$

The covariance between the project debt and the market return in (2.10) can be expressed as:

$$
\begin{aligned}
\text{Cov}(\tilde{D}_{1,p}, \tilde{R}_m) &= -d_1 \text{ Cov}(\delta_b, \tilde{R}_m) + (1-e_v)(1-b_v) \text{ Cov}(\delta_b\delta_q\widetilde{Rev}, \tilde{R}_m) - \\
&\quad -[e_f(1-b_v) + b_f] \text{ Cov}(\delta_b\delta_q, \tilde{R}_m) \qquad (2.21)
\end{aligned}
$$

Substituting the covariances on the right-hand side of the above equation by (A.21) and (A.23) (See Appendix A), and rearranging terms gives:

$$
\begin{aligned}
\text{Cov}(\tilde{D}_{1,p}, \tilde{R}_m) &= \text{Cov}(\widetilde{Rev}, \tilde{R}_m)\Big\{d_1 f_{Rev}(a'') + (1-e_v)(1-b_v)\,[F_{Rev}(a'') - F_{Rev}(b'') + \\
&\quad + b'' f_{Rev}(b'') - a'' f_{Rev}(a'')] - \\
&\quad -[e_f(1-b_v) + b_f]\left(f_{Rev}(b'') - f_{Rev}(a'')\right)\Big\} \qquad (2.22)
\end{aligned}
$$

Replacing a'' and b'' by their definitions and multiplying both sides by λ yields:

$$
\begin{aligned}
\lambda \text{ Cov}(\tilde{D}_{1,p}, \tilde{R}_m) &= \lambda \text{ Cov}(\widetilde{Rev}, \tilde{R}_m)\,\{(1-e_v)(1-b_v)\,[F_{Rev}(a'') - F_{Rev}(b'')] + \\
&\quad + [b_f + b_v d_1]\, f_{Rev}(a'')\} \qquad (2.23)
\end{aligned}
$$

Equation 2.23 shows that the systematic risk premium on the project's debt, λ Cov$(\tilde{D}_1, \tilde{R}_m)$, is equal to the systematic risk premium on the project's revenue, λ Cov$(\widetilde{Rev}, \tilde{R}_m)$, multiplied by a factor that represents the probability that debtholders would only receive some partial payment (given by the occurrence of bankruptcy), $F_{Rev}(a'') - F_{Rev}(b'')$; minus the systematic risk premium on the project's expenses, $e_v\lambda$ Cov$(\widetilde{Rev}, \tilde{R}_m)$, multiplied by the factor $F_{Rev}(a'') - F_{Rev}(b'')$; plus the systematic risk on the project's bankruptcy costs, λ Cov$(\widetilde{Rev}, \tilde{R}_m)\{[b_f + b_v d_1] f_{Rev}(a'') - b_v(1-e_v)[F_{Rev}(a'') - F_{Rev}(b'')]\}$. Note that the systematic risk premium of the project's debt is independent of the fixed expenses of the project (as these expenses occur regardless of the the level of revenues produced by the project) and is equal to zero if the project has no possibility of going bankrupt.

The market value of debt, D_p, can be calculated by substituting (2.20) and (2.23) into (2.10):

$$
\begin{aligned}
D_p &= \frac{1}{R_f}\Big\{d_1\,(1 - F_{Rev}(a'')) + (1-e_v)(1-b_v)\,\Big[E[\widetilde{Rev}]\,(F_{Rev}(a'') - F_{Rev}(b'')) + \\
&\quad + \sigma^2_{Rev}\,(f_{Rev}(b'') - f_{Rev}(a''))\Big] - [e_f(1-b_v) + b_f]\,(F_{Rev}(a'') - F_{Rev}(b'')) - \\
&\quad -\lambda \text{ Cov}(\widetilde{Rev}, \tilde{R}_m)\,\Big[(1-e_v)(1-b_v)\,[F_{Rev}(a'') - F_{Rev}(b'')] + \\
&\quad + [b_f + b_v d_1]\, f_{Rev}(a'')\Big]\Big\} \qquad (2.24)
\end{aligned}
$$

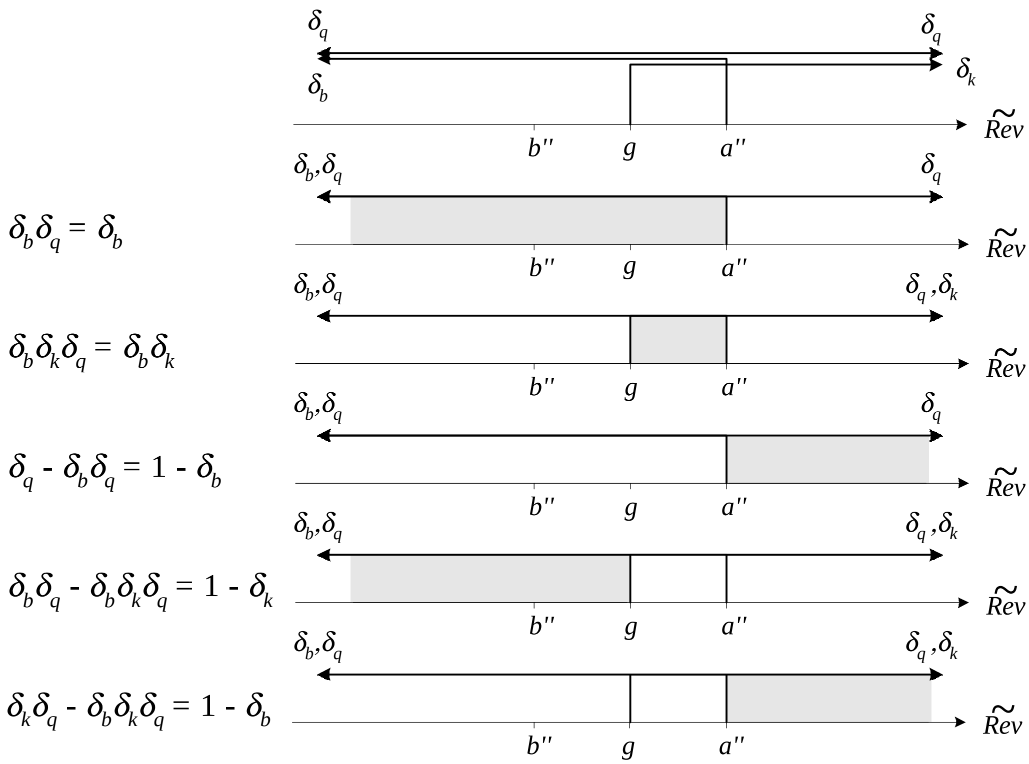

Figure 2-3: Combinations of δ_b, δ_k and δ_q When $b'' \leq$ guarantee $< a''$

The above formulation is similar to the one derived in the previous chapter. The only difference is that this formulation relates the project's revenues with the market while the previous one related the project's net income to the market. The explanation for this similarity is simple. When the guarantee is smaller than b'', it does not assure that debtholders will receive any money (*i.e.,* all the money provided by the guarantee would be used to cover bankruptcy costs). Thus, debtholders do not place any value on the guarantee and the present value of the debt remains unaltered.

$b'' \leq$ **guarantee** $< a''$

Figure 2-3 shows the values of the different combinations of δ_b, δ_k and δ_q when the production guarantee is greater than b'' but smaller than a''. Substituting these values in (2.17) gives:

$$\tilde{D}_1 = d_1(1-\delta_b) + (1-e_v)(1-b_v)[\delta_b\delta_k\widetilde{Rev} + g(1-\delta_k)] - [e_f(1-b_v) + b_f]\delta_b \qquad (2.25)$$

The expected value of the end-of-period debt, $E[\tilde{D}_1]$, can be calculated in (2.25) as:

$$E[\tilde{D}_1] = d_1(1 - E[\delta_b]) + (1 - e_v)(1 - b_v)\left\{E[\delta_b \delta_k \widetilde{Rev}] + g\,(1 - E[\delta_k])\right\} - -[e_f(1 - b_v) + b_f]E[\delta_b] \quad (2.26)$$

Substituting (A.11), (A.12), and (A.14) into (2.26) and rearranging the terms gives:

$$E[\tilde{D}_1] = d_1\,(1 - F_{Rev}(a'')) + (1 - e_v)(1 - b_v)\left\{E[\widetilde{Rev}]\,(F_{Rev}(a'') - F_{Rev}(g)) + +\sigma^2_{Rev}\,(f_{Rev}(g) - f_{Rev}(a'')) + gF_{Rev}(g)\right\} - [e_f(1 - b_v) + b_f]F_{Rev}(a'') \quad (2.27)$$

Therefore, the expected end-of-period payment to debtholders after bankruptcy costs is the full promised amount d_1 multiplied by the probability that the project does not go bankrupt; plus the conditional expected value of $\widetilde{Rev}_g$ (*i.e.*, project's revenues considering the possibility of exercising the production guarantee) given that the project is bankrupt, $E[\widetilde{Rev}]\,(F_{Rev}(a'') - F_{Rev}(g)) + \sigma^2_{Rev}\,(f_{Rev}(g) - f_{Rev}(a'')) + gF_{Rev}(g)$; minus the conditional expected value of the project expenses, $e_v[E[\widetilde{Rev}]\,(F_{Rev}(a'') - F_{Rev}(g)) +$
$+\sigma^2_{Rev}\,(f_{Rev}(g) - f_{Rev}(a'')) + gF_{Rev}(g)] + e_f F_{Rev}(a'')$; minus the expected value of the bankruptcy costs.

The covariance between the project debt and the market return can be expressed as:

$$\begin{aligned} \text{Cov}(\tilde{D}_1, \tilde{R}_m) &= \text{Cov}(d_1(1 - \delta_b) + (1 - e_v)(1 - b_v)[\delta_b \delta_k \widetilde{Rev} + g(1 - \delta_k)] - \\ &\quad -[e_f(1 - b_v) + b_f]\delta_b, \tilde{R}_m) \\ &= -d_1\ \text{Cov}(\delta_b, \tilde{R}_m) + (1 - e_v)(1 - b_v)[\ \text{Cov}(\delta_b \delta_k \widetilde{Rev}, \tilde{R}_m) - \\ &\quad -g\ \text{Cov}(\delta_k, \tilde{R}_m)] - [e_f(1 - b_v) + b_f]\ \text{Cov}(\delta_b, \tilde{R}_m) \end{aligned} \quad (2.28)$$

Substituting the covariances on the right-hand side of the above equation by (A.21) and (A.23) (See Appendix A) and rearranging terms gives:

$$\begin{aligned} \text{Cov}(\tilde{D}_1, \tilde{R}_m) &= \text{Cov}(\widetilde{Rev}, \tilde{R}_m)\left\{d_1 f_{Rev}(a'') + (1 - e_v)(1 - b_v)\,[F_{Rev}(a'') - F_{Rev}(g) + \right. \\ &\quad +g f_{Rev}(g) - a'' f_{Rev}(a'') - g f_{Rev}(g)] + \\ &\quad \left. +[e_f(1 - b_v) + b_f] f_{Rev}(a'')\right\} \end{aligned} \quad (2.29)$$

Replacing a'' and b'' by their definitions and multiplying both sides by λ yields:

$$\begin{aligned} \lambda\ \text{Cov}(\tilde{D}_1, \tilde{R}_m) &= \lambda\ \text{Cov}(\widetilde{Rev}, \tilde{R}_m)\,\{(1 - e_v)(1 - b_v)\,[F_{Rev}(a'') - F_{Rev}(g)] + \\ &\quad + [b_f + b_v d_1]\, f_{Rev}(a'')\} \end{aligned} \quad (2.30)$$

Equation 2.30 shows that the systematic risk premium on the project's debt, λ Cov$(\tilde{D}_1, \tilde{R}_m)$, is equal to the systematic risk premium on the project's revenue, λ Cov$(\widetilde{Rev}, \tilde{R}_m)$, multiplied by a factor that represents the probability that debtholders would only receive some partial payment (due to the occurrence of bankruptcy) but yet would not be able to exercise the production guarantee, $F_{Rev}(a'') - F_{Rev}(g)$; minus the systematic risk premium on the project's expenses, $e_v\lambda$ Cov$(\widetilde{Rev}, \tilde{R}_m)$, multiplied by the factor $F_{Rev}(a'') - F_{Rev}(g)$; plus the systematic risk on the project's bankruptcy costs, λ Cov$(\widetilde{Rev}, \tilde{R}_m)\{[b_f + b_v d_1] f_{Rev}(a'') - b_v(1 - e_v)[F_{Rev}(a'') - F_{Rev}(g)]\}$.

The market value of debt, D_p, can be calculated by substituting (2.27) and (2.30) into (2.10):

$$
\begin{aligned}
D_p \;=\; & \frac{1}{R_f}\Big\{d_1\left(1 - F_{Rev}(a'')\right) + (1-e_v)(1-b_v)\Big[E[\widetilde{Rev}]\left(F_{Rev}(a'') - F_{Rev}(g)\right) + \\
& + \sigma^2_{Rev}\left(f_{Rev}(g) - f_{Rev}(a'')\right) + gF_{Rev}(g)\Big] - [e_f(1-b_v) + b_f]F_{Rev}(a'') - \\
& - \lambda \text{ Cov}(\widetilde{Rev}, \tilde{R}_m)\Big[(1-e_v)(1-b_v)\left[F_{Rev}(a'') - F_{Rev}(g)\right] + \\
& + \left[b_f + b_v d_1\right] f_{Rev}(a'')\Big]\Big\} \qquad (2.31)
\end{aligned}
$$

Note that the project's debt has no systematic relationship with the market when debtholders exercise their production guarantee. This is because the guarantee offered by the government does not depend on any market conditions. Observe also that fixed expenses, e_f, always occur regardless of market conditions, hence they do not affect the systematic risk premium on the project's debt.

guarantee $\geq a''$

Figure 2-4 shows the values of the different combinations of δ_b, δ_k and δ_q when the production guarantee is greater than a''. Substituting these values in (2.17) gives:

$$\tilde{D}_{1,p} = d_1(1 - \delta_b) \qquad (2.32)$$

The expected value of $\tilde{D}_{1,p}$ is $E[\tilde{D}_{1,p}] = d_1(1 - E[\delta_b]) = d_1$ and the covariance between the project debt and the market return, Cov$(\tilde{D}_{1,p}, \tilde{R}_m)$, is zero ($\beta_{D_p} = 0$). Thus, the market value of debt, D_p, is:

$$D_p = \frac{d_1}{R_f} \qquad (2.33)$$

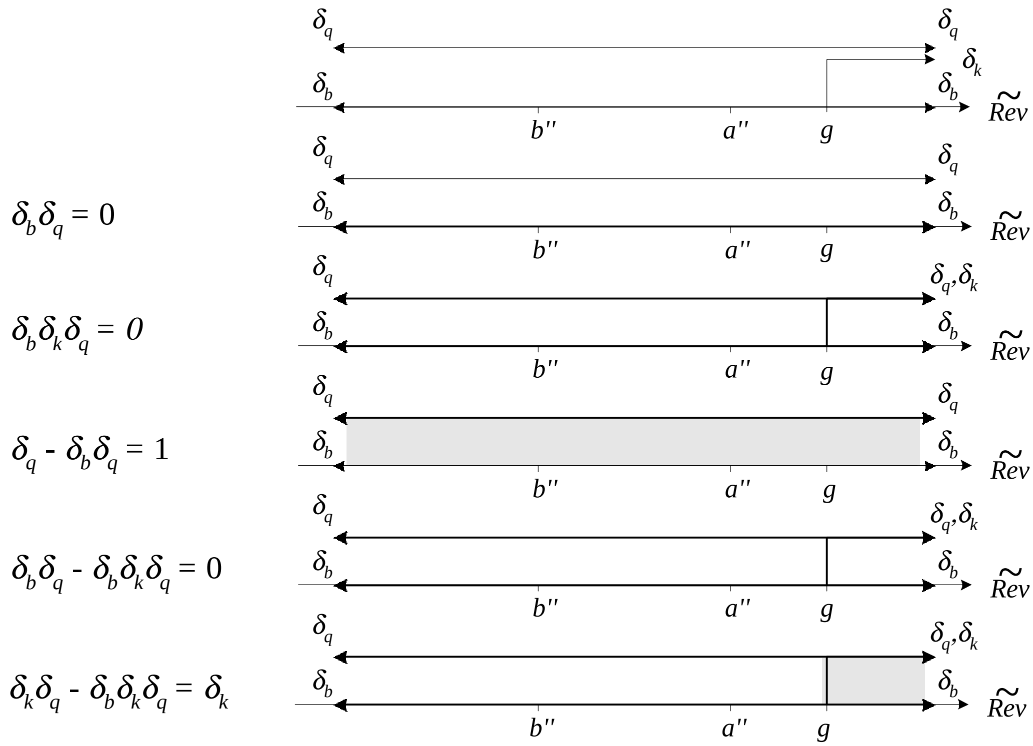

Figure 2-4: Combinations of δ_b, δ_k and δ_q When guarantee $\geq a''$

The above equation shows that the project's debt does not have any systematic relationship with the market. Because the guarantee is larger than the sum of the debt amount promised and the project expenses, the owning company would always be able to fully repay its debt regardless of how the market behaves. In this situation, the promised debt amount is discounted by the risk-free rate because the loan is riskless (debtholders will always receive their payments).

2.2.2 The Present Value of Equity

The actual market value of the equity of a project can be expressed in the same form used to express the present value of the debt of a project, that is

$$S_p = \frac{E[\tilde{S}_{1,p}] - \lambda \text{ Cov}(\tilde{S}_{1,p}, \tilde{R}_m)}{R_f} \tag{2.34}$$

where $E[\tilde{S}_{1,p}]$ is the expected value of the end-of-period value of equity when the project is backed by a guarantee revenue and $\text{Cov}(\tilde{S}_{1,p}, \tilde{R}_m)$ is the covariance between the end-of- period value of equity and the return on the market.

As shown in the previous chapter, the end-of-period value of equity represents the value of the company after debt obligations are paid to debtholders and taxes are paid to the government. Therefore, $\tilde{S}_{1,p}$ can be expressed as:

$$\tilde{S}_{1,p} = \begin{cases} (1-T)(\tilde{X}_p - d_1) + T(A-D) & \text{if } \tilde{X}_p \geq d_1 \\ 0 & \text{if } \tilde{X}_p < d_1 \end{cases} \tag{2.35}$$

or, equivalently,

$$\tilde{S}_{1,p} = \begin{cases} (1-T)[(1-e_v)\widetilde{Rev}_g - d_1 - e_f)] + T(A-D) & \text{if } \widetilde{Rev}_g \geq a'' \\ 0 & \text{if } \widetilde{Rev}_g < a'' \end{cases} \tag{2.36}$$

Alternatively, $\tilde{S}_{1,p}$ can be expressed in the following equation form:

$$\tilde{S}_{1,p} = (1-T)(\tilde{X}_p - d_1)(1-\delta_b)\delta_q + T(A-D)(1-\delta_b)\delta_q \tag{2.37}$$

Substituting $\tilde{X}_p = (1-e_v)\widetilde{Rev}_g - e_f = (1-e_v)[\delta_k\widetilde{Rev} + g(1-\delta_k)] - e_f$ into (2.37) gives:

$$\begin{aligned} \tilde{S}_{1,p} &= (1-T)\left\{(1-e_v)\left[\delta_k\widetilde{Rev} + g(1-\delta_k)\right] - e_f - d_1\right\}(1-\delta_b)\delta_q + \\ &\quad +T(A-D)(1-\delta_b)\delta_q \qquad (2.38) \\ &= (1-T)(1-e_v)\left[(\widetilde{Rev} - g)(\delta_k\delta_q - \delta_b\delta_k\delta_q)\right] + \\ &\quad +(1-T)\left(g(1-e_v) - e_f - d_1\right)(\delta_q - \delta_b\delta_q) + T(A-D)(\delta_q - \delta_b\delta_q) \qquad (2.39) \end{aligned}$$

$0 \leq$ **guarantee** $< a''$

According to Figures 2-2 and 2-3, $\delta_k\delta_q - \delta_b\delta_k\delta_q$ and $\delta_q - \delta_b\delta_q$ are equivalent to $1-\delta_b$. Thus, in this interval, $\tilde{S}_{1,p}$ can be reduced to:

$$\tilde{S}_{1,p} = (1-T)[(1-e_v)\widetilde{Rev} - d_1 - e_f](1-\delta_b) + T(A-D)(1-\delta_b) \tag{2.40}$$

The expected value of the end-of-period equity, $E[\tilde{S}_{1,p}]$, can be calculated from (2.40) as:

$$\begin{aligned} E[\tilde{S}_{1,p}] &= (1-T)[(1-e_v)E[\widetilde{Rev}] - d_1 - e_f] - (1-T)[(1-e_v)E[\delta_b\widetilde{Rev}] - \\ &\quad -d_1E[\delta_b] - e_fE[\delta_b]] + [T(A-D) - (1-T)e_f](1-E[\delta_b]) \qquad (2.41) \end{aligned}$$

The expected values on the right-hand side of (2.41) are derived in Appendix A. Substituting (A.11) and (A.14) into (2.41) yields:

$$\begin{aligned} E[\tilde{S}_{1,p}] &= \left[(1-T)\left((1-e_v)E[\widetilde{Rev}] - d_1 - e_f\right)\right](1-F_{Rev}(a'')) + \\ &+ (1-T)(1-e_v)\sigma^2_{Rev} f_{Rev}(a'') + T(A-D)(1-F_{Rev}(a'')) \end{aligned} \quad (2.42)$$

Therefore, the expected end-of-period value of the owning company after all obligations have been satisfied is the after-tax conditional expected value of the project's net operating income given that the company is not bankrupt, $(1-T)\{[E[\widetilde{Rev}](1-e_v) - e_f](1-F_{Rev}(a'')) + (1-e_v)\sigma^2_{Rev} f_{Rev}(a'')\}$; minus expected value of the amount to be paid to debtholders given that the company is not bankrupt, $d_1(1-F_{Rev}(a''))$; plus the expected value of the tax credits, $[TA + T(d_1 - D)](1-F_{Rev}(a''))$.

The covariance between the project equity and the market return in (2.34) can be expressed as:

$$\begin{aligned} \text{Cov}(\tilde{S}_{1,p}, \tilde{R}_m) &= (1-T)(1-e_v)[\text{ Cov}(\widetilde{Rev}, \tilde{R}_m) - \text{ Cov}(\delta_b \widetilde{Rev}, \tilde{R}_m)] + \\ &+ (1-T)(d_1 + e_f) \text{ Cov}(\delta_b, \tilde{R}_m) - T(A-D) \text{ Cov}(\delta_b, \tilde{R}_m) \end{aligned} \quad (2.43)$$

Substituting the covariances on the right-hand side of the above equation by (A.21) and (A.23) (See Appendix A), and rearranging the terms gives:

$$\begin{aligned} \text{Cov}(\tilde{S}_{1,p}, \tilde{R}_m) &= \text{Cov}(\widetilde{Rev}, \tilde{R}_m)\{(1-T)(1-e_v)[1-(F_{Rev}(a'') - a'' f_{Rev}(a''))] + \\ &+ [T(A-D) - (1-T)(d_1+e_f)] f_{Rev}(a'')\} \end{aligned} \quad (2.44)$$

Replacing a'' and b'' by their definitions and multiplying both sides by λ yields:

$$\begin{aligned} \lambda \text{ Cov}(\tilde{S}_{1,p}, \tilde{R}_m) &= \lambda \text{ Cov}(\widetilde{Rev}, \tilde{R}_m)\{(1-T)(1-e_v)(1-F_{Rev}(a'')) + \\ &+ T(A-D) f_{Rev}(a'')\} \end{aligned} \quad (2.45)$$

Hence, the systematic risk premium on the company's equity, given the existence of a production guarantee that is smaller than a'', $\lambda \text{ Cov}(\tilde{S}_{1,p}, \tilde{R}_m)$, is equal to the after-tax project's systematic operating risk premium, $(1-T)(1-e_v)\lambda \text{ Cov}(\widetilde{Rev}, \tilde{R}_m)$, multiplied by the probability that the company does not go bankrupt plus the systematic risk on tax credits, $T(A-D) f_{Rev}(a'')\lambda \text{ Cov}(\widetilde{Rev}, \tilde{R}_m)$.

The market value of equity, S_p, can be calculated by substituting (2.42) and (2.45) into (2.34):

$$\begin{aligned} S_p \;=\; & \frac{1}{R_f}\Big\{\Big[(1-T)\Big((1-e_v)E[\widetilde{Rev}]-d_1-e_f\Big)\Big](1-F_{Rev}(a''))+ \\ & +(1-T)(1-e_v)\sigma^2_{Rev}f_{Rev}(a'')+T(A-D)(1-F_{Rev}(a''))- \\ & -\lambda\ \mathrm{Cov}(\widetilde{Rev},\tilde{R}_m)\{(1-T)(1-e_v)(1-F_{Rev}(a''))+ \\ & +T(A-D)f_{Rev}(a'')\}\Big\} \end{aligned} \qquad (2.46)$$

Similarly to the formulation derived to determine the present value of debt, the above formulation is comparable to the one derived in the previous chapter. Thus, when the guarantee is smaller than b'', the present value of equity has the same value as if the guarantee did not exist. (In case of bankruptcy, the amount secured by the guarantee is only sufficient to pay the ‘‘liquidators.’’)

When the guarantee is larger than b'' but smaller than a'', it provides indirect value to equity-holders. That is, the guarantee does not secure any money to investors but only to debtholders. This secured debt repayment, although partial, reduces the risk faced by the lenders, hence, *ceteris paribus,* the owning company would be able to increase the amount borrowed by simply promising to pay the same repayment amount d_1 it would promise if there were no project guarantees (or guarantees were smaller than b'').

guarantee $\geq a''$

According to Figure 2-4, $\delta_k\delta_q - \delta_b\delta_k\delta_q = \delta_b$ and $\delta_q - \delta_b\delta_q = 1$. Thus, (2.39) gives:

$$\tilde{S}_{1,p} = (1-T)\Big\{(1-e_v)\Big[\delta_k\widetilde{Rev}+g(1-\delta_k)\Big]-d_1-e_f]\Big\}+T(A-D) \qquad (2.47)$$

The expected value of the end-of-period equity, $E[\tilde{S}_{1,p}]$, can be calculated in (2.47) as:

$$E[\tilde{S}_{1,p}] = (1-T)\Big\{(1-e_v)\Big[E[\delta_k\widetilde{Rev}]+g(1-E[\delta_k])\Big]-d_1-e_f]\Big\}+T(A-D) \qquad (2.48)$$

Substituting (A.12) and (A.14) into (2.48) yields:

$$\begin{aligned} E[\tilde{S}_{1,p}] \;=\; & (1-T)\Big\{(1-e_v)\Big[E[\widetilde{Rev}](1-F_{Rev}(g))+gF_{Rev}(g)+\sigma^2_{Rev}f_{Rev}(g)\Big]- \\ & -d_1-e_f\Big\}+T(A-D) \end{aligned} \qquad (2.49)$$

Hence, the expected end-of-period value of the owning company after all obligations have been satisfied is the after-tax conditional expected value of the project's net operating income given that the company does not exercise the production guarantee, $(1-T)\{[E[\widetilde{Rev}](1-e_v)-e_f](1-F_{Rev}(g))+\sigma^2_{Rev}f_{Rev}(g)\}$; plus the after-tax expected value of the guarantee, $(1-T)(gF_{Rev}(g))$; minus the debt obligations, d_1; plus tax credits, $TA+T(d_1-D)$.

The covariance between the project equity and the market return in (2.34) can be expressed as:

$$\text{Cov}(\tilde{S}_{1,p},\tilde{R}_m)=(1-T)(1-e_v)\left[\ \text{Cov}(\delta_k\widetilde{Rev},\tilde{R}_m)-g\ \text{Cov}(\delta_k,\tilde{R}_m)\right] \tag{2.50}$$

Substituting the covariances on the right-hand side of the above equation by (A.21) and (A.23), rearranging terms, and multiplying both sides by λ gives:

$$\lambda\ \text{Cov}(\tilde{S}_{1,p},\tilde{R}_m)=\lambda\ \text{Cov}(\widetilde{Rev},\tilde{R}_m)\left\{(1-T)(1-e_v)[1-F_{Rev}(g)]\right\} \tag{2.51}$$

The systematic risk premium on the company's equity, given the existence of a production guarantee that is larger than a'', $\lambda\ \text{Cov}(\tilde{S}_{1,p},\tilde{R}_m)$, is equal to the after-tax project's systematic operating risk premium, $(1-T)(1-e_v)\lambda\ \text{Cov}(\widetilde{Rev},\tilde{R}_m)$, multiplied by the probability that the company does not exercise the production guarantee.

The market value of equity, S_p, can be calculated by substituting (2.49) and (2.51) into (2.34):

$$\begin{aligned} S_p \quad = \quad & \frac{1}{R_f}\Big\{(1-T)\Big\{(1-e_v)\Big[E[\widetilde{Rev}](1-F_{Rev}(g))+gF_{Rev}(g)+\sigma^2_{Rev}f_{Rev}(g)\Big]-d_1- \\ & -e_f\Big\}+T(A-D)-\lambda\ \text{Cov}(\widetilde{Rev},\tilde{R}_m)\left\{(1-T)(1-e_v)[1-F_{Rev}(g)]\right\}\Big\} \end{aligned} \tag{2.52}$$

2.3 Minimum-Revenue Guarantee

A minimum-revenue guarantee assures a certain level of revenue despite the actual level of operation of the facility. The net operating income for a project that has a minimum-revenue guarantee is:

$$\tilde{X}_m=\begin{cases}(1-e_v)\widetilde{Rev}-e_f & \text{if } \widetilde{Rev}\geq g\\ g-e_v\widetilde{Rev}-e_f & \text{if } \widetilde{Rev}<g\end{cases} \tag{2.53}$$

Alternatively, $\tilde{X}_m$ can be expressed in the following equation form:

$$\tilde{X}_m \quad = \quad [(1-e_v)\widetilde{Rev}-e_f]\delta_k+(g-e_v\widetilde{Rev}-e_f)(1-\delta_k) \tag{2.54}$$

$$= \quad \delta_k\widetilde{Rev}-e_v\widetilde{Rev}+g(1-\delta_k)-e_f \tag{2.55}$$

where δ_k is a binary variable that was defined in (2.9).

2.3.1 The Present Value of Debt

According to the CAPM, the actual market value of debt can be computed as:

$$D_m = \frac{E[\tilde{D}_{1,m}] - \lambda \text{ Cov}(\tilde{D}_{1,m}, \tilde{R}_m)}{R_f} \tag{2.56}$$

where $E[\tilde{D}_{1,m}]$ is the expected value of the debt at time 1 when the project is backed by a minimum-revenue guarantee; λ is the market price per unit of risk; $\text{Cov}(\tilde{D}_{1,m}, \tilde{R}_m)$ is the covariance between the value of debt at time 1 and one-plus-the-rate-of-return-on-the-market; and R_f is one-plus-the-risk-free-rate.

The end-of-period value of debt, $\tilde{D}_{1,m}$, depends on the end-of-period project net operating income, $\tilde{X}_m$, and can be expressed by either of the following:

$$\tilde{D}_{1,m} = \begin{cases} d_1 & \text{if } \tilde{X}_m \geq d_1 \\ \tilde{X}_m - \tilde{B} & \text{if } \tilde{B} \leq \tilde{X}_m < d_1 \\ 0 & \text{if } \tilde{X}_m < \tilde{B} \end{cases} \tag{2.57}$$

$$\tilde{D}_{1,m} = \begin{cases} d_1 & \text{if } \widetilde{Rev}_g \geq a'' \\ (1-b_v)[\widetilde{Rev}_g - e_v\widetilde{Rev} - e_f] - b_f & \text{if } b'' \leq \widetilde{Rev}_g < a'' \\ 0 & \text{if } \widetilde{Rev}_g < b'' \end{cases} \tag{2.58}$$

Thus, if the net cash flow at the end of the period is greater than the promised amount d_1, the debtholders receive the full debt payments. Otherwise, they receive the net cash flow minus the bankruptcy costs, provided this difference is positive, and nothing if the difference is negative (the entire net operating income is consumed by bankruptcy costs).

Alternatively, $\tilde{D}_{1,m}$ can be expressed in the following equation form:

$$\tilde{D}_{1,m} = d_1(1-\delta_b)\delta_q + \delta_b\delta_q\tilde{X}_m - \delta_b\delta_q(b_f + b_v\tilde{X}_m) \tag{2.59}$$

Substituting (2.55) into (2.59) gives:

$$\begin{aligned} \tilde{D}_{1,m} &= d_1(1-\delta_b)\delta_q + \delta_b\delta_q\left(\delta_k\widetilde{Rev} - e_v\widetilde{Rev} + g(1-\delta_k) - e_f\right) - \\ &\quad -\delta_b\delta_q\left[b_f + b_v\left(\delta_k\widetilde{Rev} - e_v\widetilde{Rev} + g(1-\delta_k) - e_f\right)\right] \qquad (2.60) \\ &= d_1(1-\delta_b)\delta_q + (1-b_v)\left(\delta_b\delta_k\delta_q\widetilde{Rev} - e_v\delta_b\delta_q\widetilde{Rev} + g(\delta_b\delta_q - \delta_b\delta_k\delta_q)\right) - \\ &\quad -[e_f(1-b_v) + b_f]\delta_b\delta_q \qquad (2.61) \end{aligned}$$

$0 \leq$ **guarantee** $< b''$

Substituting the information contained in Figure 2-2 into (2.61) gives:

$$\tilde{D}_{1,m} = d_1(1-\delta_b) + (1-e_v)(1-b_v)\delta_b\delta_q\widetilde{Rev} - [e_f(1-b_v)+b_f]\delta_b\delta_q \tag{2.62}$$

Following a procedure similar to the one used in the ''production-guarantee'' section, the expected value of the end-of-period debt, $E[\tilde{D}_{1,m}]$, is:

$$\begin{aligned} E[\tilde{D}_{1,m}] &= d_1(1-E[\delta_b]) + (1-e_v)(1-b_v)E[\delta_b\delta_q\widetilde{Rev}] - [e_f(1-b_v)+b_f]E[\delta_b\delta_q] \qquad (2.63) \\ &= d_1\left(1-F_{Rev}(a'')\right) + (1-e_v)(1-b_v)\Big\{E[\widetilde{Rev}]\left(F_{Rev}(a'')-F_{Rev}(b'')\right)+ \\ &\quad +\sigma^2_{Rev}\left(f_{Rev}(b'')-f_{Rev}(a'')\right)\Big\} - \\ &\quad -[e_f(1-b_v)+b_f]\left(F_{Rev}(a'')-F_{Rev}(b'')\right) \qquad (2.64) \end{aligned}$$

and the covariance between the project debt and the market return is:

$$\begin{aligned} \text{Cov}(\tilde{D}_{1,m},\tilde{R}_m) &= -d_1\ \text{Cov}(\delta_b,\tilde{R}_m) + (1-e_v)(1-b_v)\ \text{Cov}(\delta_b\delta_q\widetilde{Rev},\tilde{R}_m) - \\ &\quad -[e_f(1-b_v)+b_f]\ \text{Cov}(\delta_b\delta_q,\tilde{R}_m) \qquad (2.65) \\ &= \text{Cov}(\widetilde{Rev},\tilde{R}_m)\left\{(1-e_v)(1-b_v)\left[F_{Rev}(a'')-F_{Rev}(b'')\right]+\right. \\ &\quad \left.+\left[b_f+b_vd_1\right]f_{Rev}(a'')\right\} \qquad (2.66) \end{aligned}$$

Note that (2.20) and (2.64), (2.23) and (2.66) are identical. Thus, the present value of debt for a project that has either a production or a minimum-revenue guarantee smaller than b'' is equal to the market value of debt when there are no guarantees and is given by (2.24).

$b'' \leq$ **guarantee** $< a''$

Substituting the information contained in Figure 2-3 into (2.61) gives:

$$\tilde{D}_{1,m} = d_1(1-\delta_b) + (1-b_v)\left(\delta_b\delta_k\widetilde{Rev} - e_v\delta_b\widetilde{Rev} + g(1-\delta_k)\right) - [e_f(1-b_v)+b_f]\delta_b \tag{2.67}$$

Following a procedure similar to the one used to calculate $E[\tilde{D}_{1,p}]$ and $\text{Cov}(\tilde{D}_{1,p},\tilde{R}_m)$, the expected value of the end-of-period debt, $E[\tilde{D}_{1,m}]$, can be calculated in (2.67) as:

$$\begin{aligned} E[\tilde{D}_{1,m}] &= d_1(1-E[\delta_b]) + (1-b_v)\Big\{E[\delta_b\delta_k\widetilde{Rev}] - e_vE[\delta_b\widetilde{Rev}] + g(1-E[\delta_k])\Big\} \\ &\quad -[e_f(1-b_v)+b_f]E[\delta_b] \qquad (2.68) \end{aligned}$$

$$
\begin{aligned}
&= d_1\left(1-F_{Rev}(a'')\right)+(1-b_v)\Big\{E[\widetilde{Rev}]\left(F_{Rev}(a'')-F_{Rev}(g)\right)+ \\
&\quad +\sigma^2_{Rev}\left(f_{Rev}(g)-f_{Rev}(a'')\right)-e_v[E[\widetilde{Rev}]F_{Rev}(a'')-\sigma^2_{Rev}f_{Rev}(a'')]+ \\
&\quad +gF_{Rev}(g)\Big\}-[e_f(1-b_v)+b_f]F_{Rev}(a'') \qquad (2.69)\\
&= d_1\left(1-F_{Rev}(a'')\right)+(1-e_v)(1-b_v)\Big\{E[\widetilde{Rev}]\left(F_{Rev}(a'')-F_{Rev}(g)\right)+ \\
&\quad +\sigma^2_{Rev}\left(f_{Rev}(g)-f_{Rev}(a'')\right)+gF_{Rev}(g)\Big\}- \\
&\quad -[e_f(1-b_v)+b_f]F_{Rev}(a'')+(1-b_v)e_v[gF_{Rev}(g)- \\
&\quad -E[\widetilde{Rev}]F_{Rev}(g)+\sigma^2_{Rev}f_{Rev}(g)] \qquad (2.70)
\end{aligned}
$$

and the covariance between the project debt and the market return is:

$$
\begin{aligned}
\text{Cov}(\tilde{D}_{1,m},\tilde{R}_m) &= -d_1\ \text{Cov}(\delta_b,\tilde{R}_m)+(1-b_v)\Big\{\ \text{Cov}(\delta_b\delta_k\widetilde{Rev},\tilde{R}_m)- \\
&\quad -e_v\ \text{Cov}(\delta_b\widetilde{Rev},\tilde{R}_m)-g\ \text{Cov}(\delta_k,\tilde{R}_m)\Big\}- \\
&\quad -[e_f(1-b_v)+b_f]\ \text{Cov}(\delta_b,\tilde{R}_m) \qquad (2.71)\\
&= \text{Cov}(\widetilde{Rev},\tilde{R}_m)\{(1-e_v)(1-b_v)\left[F_{Rev}(a'')-F_{Rev}(g)\right]+ \\
&\quad +\left[b_f+b_vd_1\right]f_{Rev}(a'')-e_vF_{Rev}(g)\} \qquad (2.72)
\end{aligned}
$$

Equation 2.69 shows that the expected end-of-period payment to debtholders after bankruptcy costs is the full promised amount d_1 multiplied by the probability that the project does not go bankrupt; plus the conditional expected value of $\widetilde{Rev}_g$ given that the project is bankrupt, $E[\widetilde{Rev}](F_{Rev}(a'')-F_{Rev}(g))+\sigma^2_{Rev}\left(f_{Rev}(g)-f_{Rev}(a'')\right)+gF_{Rev}(g)$; minus the conditional expected value of the project expenses, $e_v[E[\widetilde{Rev}]F_{Rev}(a'')-\sigma^2_{Rev}f_{Rev}(a'')]+e_fF_{Rev}(a'')$; minus the conditional expected value of the bankruptcy costs, $b_v\{E[\widetilde{Rev}]F_{Rev}(a'')-F_{Rev}(g)+\sigma^2_{Rev}f_{Rev}(g)-f_{Rev}(a'')\}+(b_f-e_fb_v)F_{Rev}(a'')$.

Equation 2.72 shows that the systematic risk premium on the project's debt, λ Cov$(\tilde{D}_1,\tilde{R}_m)$, is equal to the systematic risk premium on the project's revenue, λ Cov$(\widetilde{Rev},\tilde{R}_m)$, multiplied by a factor that represents the probability that debtholders would only receive some partial payment (due to the occurrence of bankruptcy) but yet would not be able to exercise the production guarantee, $F_{Rev}(a'')-F_{Rev}(g)$; minus the systematic risk premium on the project's expenses, $e_v\lambda$ Cov$(\widetilde{Rev},\tilde{R}_m)$, multiplied by the probability the project goes bankrupt, $F_{Rev}(a'')$; plus the systematic risk on the project's bankruptcy costs, λ Cov$(\widetilde{Rev},\tilde{R}_m)\{[b_f+b_vd_1]f_{Rev}(a'')-b_v(1-e_v)[F_{Rev}(a'')-F_{Rev}(g)]\}$.

The market value of debt, D_m, can be calculated by substituting (2.69) and (2.72) into (2.56):

$$\begin{aligned} D_m &= \frac{1}{R_f}\Big\{d_1\,(1 - F_{Rev}(a'')) + (1 - b_v)\Big\{E[\widetilde{Rev}]\,(F_{Rev}(a'') - F_{Rev}(g)) + \\ &+ \sigma^2_{Rev}\,(f_{Rev}(g) - f_{Rev}(a'')) - e_v[E[\widetilde{Rev}]F_{Rev}(a'') - \sigma^2_{Rev} f_{Rev}(a'')] + \\ &+ gF_{Rev}(g)\Big\} - [e_f(1 - b_v) + b_f]F_{Rev}(a'') - \\ &- \lambda\ \text{Cov}(\widetilde{Rev}, \tilde{R}_m)\{(1 - e_v)(1 - b_v)\,[F_{Rev}(a'') - F_{Rev}(g)] + \\ &+ [b_f + b_v d_1]\, f_{Rev}(a'') - e_v F_{Rev}(g)\}\Big\} \end{aligned} \tag{2.73}$$

Subtracting (2.31) from (2.73) gives:

$$\begin{aligned} D_m - D_p &= \frac{1}{R_f}\Big\{(1 - b_v)e_v\Big[gF_{Rev}(g) - E[\widetilde{Rev}]F_{Rev}(g) + \sigma^2_{Rev} f_{Rev}(g)\Big] + \\ &+ \lambda\ \text{Cov}(\widetilde{Rev}, \tilde{R}_m)e_v F_{Rev}(g)\Big\} \end{aligned} \tag{2.74}$$

Therefore,when $g \geq E[\widetilde{Rev}] - \lambda\ \text{Cov}(\widetilde{Rev}, \tilde{R}_m)/(1 - b_v) - \sigma^2_{Rev} f_{Rev}(g)/F_{Rev}(g)$, $D_m - D_p$ is positive, it indicates that, for debtholders, a minimum-revenue guarantee is more valuable than a production-guarantee. If $g < E[\widetilde{Rev}] - \lambda\ \text{Cov}(\widetilde{Rev}, \tilde{R}_m)/(1 - b_v) - \sigma^2_{Rev} f_{Rev}(g)/F_{Rev}(g)$, then $D_m - D_p < 0$ and debtholders would prefer a production guarantee.

guarantee $\geq a''$

Substituting the information contained in Figure 2-4 into (2.61) yields:

$$\tilde{D}_{1,m} = d_1(1 - \delta_b) \tag{2.75}$$

Thus, the expected value of $\tilde{D}_{1,m}$ is $E[\tilde{D}_{1,m}] = d_1(1 - E[\delta_b]) = d_1$, the covariance between the project debt and the market return, $\text{Cov}(\tilde{D}_{1,m}, \tilde{R}_m)$, is zero, and $D_m = D_p = d_1/R_f$.

2.3.2 The Present Value of Equity

The actual market value of the equity of a project can be expressed in the same form used to express the market value of the debt of a project, that is

$$S_m = \frac{E[\tilde{S}_{1,m}] - \lambda\ \text{Cov}(\tilde{S}_{1,m}, \tilde{R}_m)}{R_f} \tag{2.76}$$

where $E[\tilde{S}_{1,m}]$ is the expected value of the end-of-period value of equity when the project is backed by a minimum-revenue guarantee revenue and $\text{Cov}(\tilde{S}_{1,m}, \tilde{R}_m)$ is the covariance between the end-of- period value of equity and the return on the market.

Similar to (2.37), $\tilde{S}_{1,m}$ can be expressed as:

$$\tilde{S}_{1,m} = (1-T)(\tilde{X}_m - d_1)(1-\delta_b)\delta_q + T(A-D)(1-\delta_b)\delta_q \tag{2.77}$$

and substituting $\tilde{X}_m = [(1-e_v)\widetilde{Rev} - e_f]\delta_k + (g - e_v\widetilde{Rev} - e_f)(1-\delta_k)$ into (2.77) gives:

$$\begin{aligned} \tilde{S}_{1,m} &= (1-T)\left[(\widetilde{Rev} - g)(\delta_k\delta_q - \delta_b\delta_k\delta_q) - e_v\widetilde{Rev}(\delta_q - \delta_b\delta_q)\right] + \\ &\quad +(1-T)(g - e_f - d_1)(\delta_q - \delta_b\delta_q) + T(A-D)(\delta_q - \delta_b\delta_q) \end{aligned} \tag{2.78}$$

$0 \leq$ **guarantee** $< a''$

Substituting the information contained in Figures 2-2 and 2-3 into (2.78) gives:

$$\tilde{S}_{1,m} = (1-T)\left\{(1-e_v)\widetilde{Rev} - e_f - d_1\right\}(1-\delta_b) + T(A-D)(1-\delta_b) \tag{2.79}$$

Following a procedure similar to the one used in the''production-guarantee'' section, the expected value of the end-of-period equity, $E[\tilde{S}_{1,m}]$, is:

$$\begin{aligned} E[\tilde{S}_{1,m}] &= (1-T)[(1-e_v)E[\widetilde{Rev}] - d_1] - (1-T)[(1-e_v)E[\delta_b\widetilde{Rev}] - d_1E[\delta_b]] + \\ &\quad +[T(A-D) - (1-T)e_f](1-E[\delta_b]) \qquad (2.80) \\ &= \left[(1-T)\left((1-e_v)E[\widetilde{Rev}] - d_1 - e_f\right)\right](1 - F_{Rev}(a'')) + \\ &\quad +(1-T)(1-e_v)\sigma^2_{Rev} f_{Rev}(a'') + T(A-D)(1 - F_{Rev}(a'')) \qquad (2.81) \end{aligned}$$

and the covariance between the project equity and the market return is:

$$\begin{aligned} \text{Cov}(\tilde{S}_{1,m}, \tilde{R}_m) &= (1-T)(1-e_v)[\text{ Cov}(\widetilde{Rev}, \tilde{R}_m) - \text{ Cov}(\delta_b\widetilde{Rev}, \tilde{R}_m)] + \\ &\quad +(1-T)(d_1 + e_f)\text{ Cov}(\delta_b, \tilde{R}_m) - T(A-D)\text{ Cov}(\delta_b, \tilde{R}_m) \qquad (2.82) \\ &= \text{ Cov}(\widetilde{Rev}, \tilde{R}_m)\{(1-T)(1-e_v)(1 - F_{Rev}(a'')) + \\ &\quad +T(A-D)f_{Rev}(a'')\} \qquad (2.83) \end{aligned}$$

Note that (2.42) and (2.81), (2.45) and (2.83) are identical. Thus, the market value of equity for a project that has either a production or a minimum-revenue guarantee smaller than a'' is equal to the present value of equity when there are no guarantees and is given by (2.46).

guarantee $\geq a''$

Substituting the information contained in Figure 2-4 into (2.78) gives:

$$\tilde{S}_{1,m} = (1-T)\left\{\delta_k \widetilde{Rev} + g(1-\delta_k) - e_v \widetilde{Rev} - d_1 - e_f]\right\} + T(A-D) \tag{2.84}$$

Following a procedure similar to the one used to calculate $E[\tilde{S}_{1,p}]$ and $\text{Cov}(\tilde{S}_{1,p}, \tilde{R}_m)$, the expected value of the end-of-period equity, $E[\tilde{S}_{1,m}]$, can be calculated in (2.84) as:

$$\begin{aligned} E[\tilde{S}_{1,m}] &= (1-T)\left\{E[\widetilde{Rev}](1-F_{Rev}(g)) + gF_{Rev}(g) + \sigma^2_{Rev} f_{Rev}(g) - e_v E[\widetilde{Rev}] - \right. \\ &\left. -d_1 - e_f\right\} + T(A-D) \end{aligned} \tag{2.85}$$

and the covariance between the project equity and the market return is:

$$\begin{aligned} \text{Cov}(\tilde{S}_{1,m}, \tilde{R}_m) &= (1-T)\Big[\ \text{Cov}(\delta_k \widetilde{Rev}, \tilde{R}_m) - g\ \text{Cov}(\delta_k, \tilde{R}_m) \\ &\quad -e_v\ \text{Cov}(\widetilde{Rev}, \tilde{R}_m)\Big] \qquad (2.86) \\ &= \text{Cov}(\widetilde{Rev}, \tilde{R}_m)\left\{(1-T)[1 - F_{Rev}(g) - e_v]\right\} \qquad (2.87) \end{aligned}$$

Equation 2.85 shows that the expected end-of-period value of the owning company after all obligations have been satisfied is the after-tax conditional expected value of the project revenues given that the company does not exercise the production guarantee, $(1-T)\{E[\widetilde{Rev}](1-F_{Rev}(g)) + \sigma^2_{Rev} f_{Rev}(g)\}$; plus the after-tax expected value of the guarantee, $(1-T)(gF_{Rev}(g))$; minus the after-tax expected value of the project expenses, $(1-T)\{e_v E[\widetilde{Rev}] - e_f\}$ minus the debt obligations, d_1; plus tax credits, $TA + T(d_1 - D)$.

The market value of equity, S_m, can be calculated by substituting (2.85) and (2.87) into (2.78):

$$\begin{aligned} S_m &= \frac{1}{R_f}\left\{(1-T)\left\{E[\widetilde{Rev}](1-F_{Rev}(g)) + gF_{Rev}(g) + \sigma^2_{Rev} f_{Rev}(g) - e_v E[\widetilde{Rev}] - \right.\right. \\ &\left.\left. -d_1 - e_f\right\} + T(A-D) - \lambda\ \text{Cov}(\widetilde{Rev}, \tilde{R}_m)\left\{(1-T)[1-F_{Rev}(g) - e_v]\right\}\right\} \end{aligned} \tag{2.88}$$

Subtracting (2.52) from (2.88) gives:

$$\begin{aligned} S_m - S_p &= \frac{1}{R_f}\left\{(1-T)e_v\left[gF_{Rev}(g) - E[\widetilde{Rev}]F_{Rev}(g) + \sigma^2_{Rev} f_{Rev}(g)\right] + \right. \\ &\left. +\lambda\ \text{Cov}(\widetilde{Rev}, \tilde{R}_m)(1-T)e_v F_{Rev}(g)\right\} \end{aligned} \tag{2.89}$$

Therefore,when $g \geq E[\widetilde{Rev}] - \lambda\ \text{Cov}(\widetilde{Rev}, \tilde{R}_m) - \sigma^2_{Rev} f_{Rev}(g)/F_{Rev}(g)$, $S_m - S_p$ is positive indicating that, for equityholders, a minimum-revenue guarantee is more valuable than a production-guarantee. If $g < E[\widetilde{Rev}] - \lambda\ \text{Cov}(\widetilde{Rev}, \tilde{R}_m) - \sigma^2_{Rev} f_{Rev}(g)/F_{Rev}(g)$, then $S_m - S_p < 0$ and equityholders would prefer a production guarantee.

2.4 Project Debt Capacity

The previous chapter has shown that, under the possibility of costly bankruptcy, there is a finite limit on the amount of debt a privately-financed project can accommodate. This is the case, of course, if there exists a promised amount, d_1^c, that maximizes D, and if this maximum value is smaller than the initial cost of the project (*i.e.,* $D^c = D_{\max}$ if $D_{\max} < A$). This section investigates how production and minimum-revenue guarantees influence this limit.

Let us start the analysis by examining the case of a project with a production guarantee smaller than b''. In order to prove that d_1^c exists, it is necessary to show that $\partial D/\partial d_1 = 0$ and $\partial^2 D/\partial d_1^2 < 0$. The first derivative is given by differentiating (2.24) with respect to d_1:

$$\begin{aligned} \left.\frac{\partial D}{\partial d_1}\right|_{g(p)<b''} &= \frac{1}{R_f}\left\{\frac{\partial d_1}{\partial d_1} - \frac{\partial[d_1 F_{Rev}(a'')]}{\partial d_1} + (1-e_v)(1-b_v)\left[E[\widetilde{Rev}]\frac{\partial F_{Rev}(a'')}{\partial d_1} - \right.\right. \\ &\left. -\sigma^2_{Rev}\frac{\partial f_{Rev}(a'')}{\partial d_1}\right] - [e_f(1-b_v)+b_f]\frac{\partial F_{Rev}(a'')}{\partial d_1} - \\ &-\lambda\ \text{Cov}(\widetilde{Rev}, \tilde{R}_m)\left[(1-e_v)(1-b_v)\frac{\partial F_{Rev}(a'')}{\partial d_1} + \right. \\ &\left.\left. +b_f\frac{\partial f_{Rev}(a'')}{\partial d_1} + b_v\frac{\partial[d_1 f_{Rev}(a'')]}{\partial d_1}\right]\right\} \end{aligned} \tag{2.90}$$

Given that $\frac{\partial F_{Rev}(a'')}{\partial d_1} = \frac{f_{Rev}(a'')}{1-e_v}$ and $\frac{\partial f_{Rev}(a'')}{\partial d_1} = -(\frac{a''-E[\widetilde{Rev}]}{\sigma^2_{Rev}})\frac{f_{Rev}(a'')}{1-e_v}$, (2.90) becomes:

$$\begin{aligned} \left.\frac{\partial D}{\partial d_1}\right|_{g(p)<b''} &= \frac{1}{R_f}\left\{1 - F_{Rev}(a'') - (b_f + b_v d_1)\frac{f_{Rev}(a'')}{1-e_v} - \right. \\ &-\lambda\ \text{Cov}(\widetilde{Rev}, \tilde{R}_m)\frac{f_{Rev}(a'')}{1-e_v}\left[1 - e_v - \right. \\ &\left.\left. -(\frac{a'' - E[\widetilde{Rev}]}{\sigma^2_{Rev}})(b_f + b_v d_1)\right]\right\} \end{aligned} \tag{2.91}$$

The second derivative, $\partial^2 D/\partial d_1^2$, can be calculated by differentiating the above equation with respect to d_1:

$$\left.\frac{\partial^2 D}{\partial d_1^2}\right|_{g(p)<b''} = \frac{f_{Rev}(a'')}{R_f(1-e_v)^2}\left\{-(1+b_v)(1-e_v)) + \left(\frac{a''-E[\widetilde{Rev}]}{\sigma^2_{Rev}}\right)\left[(b_f+b_vd_1)+ \right.\right.$$
$$+\lambda \text{ Cov}(\widetilde{Rev},\tilde{R}_m)\left((1+b_v)(1-e_v)-\right.$$
$$\left.\left.\left.-(b_f+b_vd_1)\left(\frac{a''-E[\widetilde{Rev}]}{\sigma^2_{Rev}}-\frac{1}{a''-E[\widetilde{Rev}]}\right)\right)\right]\right\} \tag{2.92}$$

The second derivative is clearly negative for any $a'' < E[\widetilde{Rev}]$ (or equivalently, $d_1 < E[\widetilde{Rev}](1-e_v) - e_f$). Thus, as long as there exists a promised amount d_1^c that satisfies $\partial D/\partial d_1 = 0$, then d_1^c corresponds to a maximum. Note that at low values of d_1, $\partial D/\partial d_1 > 0$ and values of d_1 that satisfy $\partial D/\partial d_1 < 0$ assure that $\partial D/\partial d_1 = 0$ at some finite point. Setting (2.91) to be smaller than zero and rearranging terms gives:

$$b_f + b_v d_1 > \frac{1-F_{Rev}(a'')-\lambda \text{ Cov}(\widetilde{Rev},\tilde{R}_m)f_{Rev}(a'')}{\frac{f_{Rev}(a'')}{1-e_v}\left[1-(\frac{a''-E[\widetilde{Rev}]}{\sigma^2_{Rev}})\lambda \text{ Cov}(\widetilde{Rev},\tilde{R}_m)\right]} \tag{2.93}$$

Therefore, if the above condition is satisfied, there is a finite limit on the owning company's ability to raise debt. This maximum debt amount, $D_{\max}$, is calculated by substituting the promised amount d_1^c that satisfies $\partial D/\partial d_1 = 0$ into (2.24). If $D_{\max} < A$ then project debt capacity exists(i.e., $D^c = D_{\max}$).

Using the above procedure for the case of a minimum-revenue guarantee smaller than b'' yields a formulation that is exactly the same as the one derived above. The maximum debt promised amount, d_1^c, is the same for both cases and for the more basic situation where there are no guarantees. This shows that **guarantees (either production or minimum-revenue) smaller than b'' have no value to debtholders because the guarantees do not provide any effective collateral to the loan. Therefore, a project with such guarantees has the same d_1^c and the same debt capacity as a project without any guarantees.**

The present value of the debt, when either a production or a minimum-revenue guarantee is larger than b'', is determined by two different functions.

If $d_1 \le d_1^g = g(1-e_v) - e_f$ then the debt is considered riskless. This is because the full loan amount (and the project expenses) is collaterized, hence debtholders face no repayment risks and the project has no possibility of going bankrupt. In this region, the guarantee is always equal or greater than a''. Thus, D increases as d_1 increases, and $D^g_{\max}$, the maximum amount of ‘‘riskless’’ debt that can be raised due to the presence of a guarantee always occurs when d_1 assumes its maximum possible value (i.e., when $d_1 = d_1^g$) and is equal to $D^g_{\max} = [g(1-e_v) - e_f]/R_f$.[1] If $d_1 > d_1^g$ and $g \geq [b_f + e_f(1-b_v)]/[(1-b_v)(1-e_v)]$ (i.e., $b'' \leq$ guarantee $< a''$) then $D^s_{\max}$ is calculated using (2.31) or (2.73) for, respectively, a production or a minimum-revenue guarantee. The value of d_1 that maximizes debt capacity in this case is d_1^s. **Notice that d_1^s is not a function of g.** The amount of debt that can be raised to fund the project is the larger of the two $D_{\max}$ values, that is

$$D_{\max} = \max(D^g_{\max}, D^s_{\max}) \tag{2.95}$$

Thus, d_1^c is either d_1^g or d_1^s. Again, if $D_{\max} < A$ then project debt capacity exists (i.e., $D^c = D_{\max}$) otherwise the concept is not operationalized and 100% debt financing is possible.

Table 2-1 illustrates how to determine the debt capacity of a project under different values of g and d_1 so that $D_{\max}$ for both functions are considered. It shows, for instance, that when the guarantee is larger than b'' and $D^s_{\max} < A$ are both satisfied then depending on the guarantee amount D^c can either be $D^g_{\max}$ or $D^s_{\max}$. If $D^c = D^s_{\max}$, the maximum amount of debt an owning company can promise to pay at the end of the period to debtholders is exactly the amount it would have promised to pay if the project had no guarantees. This is because $d_1^c = d_1^s$ for guarantees smaller than a'', and d_1^s is a function of project parameters (i.e., $E[\widetilde{Rev}]$, σ^2_{Rev}, e_f, e_v, ρ_{Rev,R_m}, b_f, and b_v) and market parameters (i.e., $E[\tilde{R}_m]$, σ^2_m, T, and R_f). It does not depend on the guarantee amount g. If $D^c = D^g_{\max}$, then $D^c = \{g(1-e_v) - e_f\}/R_f$.

Note that for any guarantee amount smaller than a'', d_1^s remains constant while debt capacity does not. That is, **an increase in the guarantee amount (provided $b'' < g < a''$) results in an increase in the debt capacity of the project (D^c) but does not change the promised debt amount d_1^s. This is because an increase in the guarantee decreases the risk faced**

[1] The maximum amount of a guarantee necessary to secure 100% debt financing is:

$$g_{\max} = \frac{AR_f + e_f}{1 - e_v} \tag{2.94}$$

In this extreme case, $g_{\max}$ is equal to a'', hence the present value of debt is only determined by the function $D = d_1/R_f$ and $D^g_{\max} = A$ occurs when $d_1 = g_{\max}(1-e_v) - e_f = AR_f$. Therefore, a project that has a guarantee equal to $g_{\max}$ can be 100% debt financed and does not have ‘‘debt capacity’’ as an operational concept. A guarantee greater than $g_{\max}$ does not provide any extra value to debtholders, all the extra amount of the guarantee (i.e., over $g_{\max}$) goes to equityholders and the government (e.g., taxes).

Table 2-1: Project debt capacity as a function of g and d_1

Riskless Debt (*i.e.*, $d_1 \leq d_1^g \Leftrightarrow a'' \leq g$) (1)	Risky Debt (*i.e.*, $d_1 > d_1^g \Leftrightarrow g < a''$) (2)	Project Debt Capacity D^c (3)
if $g < g_{100}$ then $D^g_{\max} = \frac{1}{R_f}[g(1-e_v) - e_f]$	if $g < b''$ then $D^s_{\max}$ given by (2.24) if $b'' \leq g$ then $D^s_{\max}$ given by (2.31)	$D^c = \max(D^g_{max}, D^s_{max})$ (if $D^s_{\max} \geq A$ then 100% debt fin. possible)
if $g \geq g_{100}$ then $D^g_{\max} = A$		100% debt fin. possible

by debtholders and hence, decreases the interest they charge on the loan even though the promised debt amount d_1^s remains the same. When $g \geq a''$ the guarantee provides full collateral for the debt, therefore, if d_1 remains constant, then an increase in the guarantee beyond a'' does not increase the debt capacity of the project. This implies that as g increases d_1 should also increase so that $g = a''$. Thus, $d_1^c = d_1^g = g(1-e_v) - e_f$ and $D_c = D^g_{\max} = d_1^g / R_f$. In this case, increasing g, increases both d_1^c and D_c.

2.5 Optimal Capital Structure

The previous chapter has also shown that the optimal financial structure always requires less debt financing than the project debt capacity. This section examines how production and minimum-revenue guarantees influence the optimal financial structure of a privately financed project.

Let us start the analysis by examining the financial objective of maximizing the return on the equityholders' investment (*ROE*) for a project with a production guarantee smaller than b'' (in this interval, a minimum-revenue guarantee yields the same formulation as a production guarantee). Differentiating $\dfrac{E[\tilde{S}_1]}{A-D}$ with respect to d_1 gives:

$$\left.\frac{\partial E[\widetilde{ROE}]}{\partial d_1}\right|_{g(p)<b''} = \frac{\partial}{\partial d_1}\left(\frac{E[\tilde{S}_1]}{(A-D)}\right) = \frac{\frac{\partial E[\tilde{S}_1]}{\partial d_1}(A-D) + E[\tilde{S}_1]\frac{\partial D}{\partial d_1}}{(A-D)^2} \tag{2.96}$$

where $\partial E[\tilde{S}_1]/\partial d_1$ is obtained by differentiating (2.42) with respect to d_1:

$$\begin{aligned}
\left.\frac{\partial E[\tilde{S}_1]}{\partial d_1}\right|_{g(p)<b''} &= -(1-T)\frac{\partial d_1}{\partial d_1}(1-F_{Rev}(a''))- \\
&-(1-T)\left\{(1-e_v)E[\widetilde{Rev}]-d_1-e_f\right\}\frac{\partial F_{Rev}(a'')}{\partial d_1}+ \\
&+(1-T)(1-e_v)\sigma^2_{Rev}\frac{\partial f_{Rev}(a'')}{\partial d_1}- \\
&-T\frac{\partial D}{\partial d_1}(1-F_{Rev}(a''))-T(A-D)\frac{\partial F_{Rev}(a'')}{\partial d_1} \qquad (2.97)\\
&= -(1-T)(1-F_{Rev}(a''))- \\
&-T\left\{(A-D)\frac{f_{Rev}(a'')}{1-e_v}+\frac{\partial D}{\partial d_1}(1-F_{Rev}(a''))\right\} \qquad (2.98)
\end{aligned}$$

Substituting (2.98) into (2.96) gives:

$$\begin{aligned}
\left.\frac{\partial E[\widetilde{ROE}]}{\partial d_1}\right|_{g(p)<b''} &= \frac{1}{(A-D)^2}\left\{(A-D)\left[-(1-T)(1-F_{Rev}(a''))-\right.\right. \\
&\left.-T\left((A-D)\frac{f_{Rev}(a'')}{1-e_v}+\frac{\partial D}{\partial d_1}(1-F_{Rev}(a''))\right)\right]+ \\
&\left.+E[\tilde{S}_1]\frac{\partial D}{\partial d_1}\right\} \qquad (2.99)
\end{aligned}$$

As the optimal capital structure occurs when $\partial E[\widetilde{ROE}]/\partial d_1 = 0$, (2.99) yields:

$$\begin{aligned}
E[\tilde{S}_1]\frac{\partial D}{\partial d_1} &= (A-D)\left\{(1-T)(1-F_{Rev}(a''))+\right. \\
&\left.+T\left[(A-D)\frac{f_{Rev}(a'')}{1-e_v}+\frac{\partial D}{\partial d_1}(1-F_{Rev}(a''))\right]\right\} \qquad (2.100)
\end{aligned}$$

and solving for $\partial D/\partial d_1$ gives:

$$\left.\frac{\partial D}{\partial d_1}\right|_{g(p)<b''} = \frac{(1-T)(1-F_{Rev}(a''))+T(A-D)\frac{f_{Rev}(a'')}{1-e_v}}{\frac{E[\tilde{S}_1]}{A-D}-T(1-F_{Rev}(a''))} \qquad (2.101)$$

The numerator and the denominator of (2.101) are positive for $d_1 < d_1^c$ and $A > D$. (See Section 1.6 for the proof of the above statement.) Consequently, when $\partial E[\widetilde{ROE}]/\partial d_1 = 0$ we always have $\partial D/\partial d_1 > 0$. Thus, similarly to the case of non-guarantees, the optimal capital structure for the concession project always occurs before its debt capacity is reached, that is, $d_1^{ROE} < d_1^c$, where d_1^{ROE} is the value of d_1 that satisfies (2.101). Therefore, **a company that has a privately financed project with a guarantee smaller than b'' and with $A > D^c$ should not borrow at the debt capacity but only at the amount sufficient to maximize its return on**

investment. If the company's debt capacity allows 100% debt financing (*i.e.*, $D^c = A$), (2.101) gives $\partial D/\partial d_1 = 0$ and the optimal capital structure occurs at 100% debt financing.

Moreover, the d_1^{ROE} and D^{ROE} of a project with a guarantee smaller than b'' are equal to the d_1^{ROE} and D^{ROE} of a project without a guarantee. This indicates that guarantees (either production or minimum-revenue) smaller than b'' have no value to investors as they do not guarantee any repayment either to equityholders or to debtholders.

The objective of maximizing the project's net present value for a project with a production guarantee smaller than b'' only provides an ''optimal capital structure'' similar to the objective of maximizing the $E[\widetilde{ROE}]$ when $E[\widetilde{ROE}] \approx R_f$. (See Section 1.6 for the proof of the above statement.) Since $E[\widetilde{ROE}] > R_f$ should always be true, the maximization of the return on equity investment always allows more borrowing than the maximization of the project's net present value.

Similar to the previous section, in order to determine the maximum expected return to project investors, $E[\widetilde{ROE}]_{\max}$, when either a production or a minimum-revenue guarantee is larger than b'', it is necessary to investigate the expected return of two different intervals: $0 < d_1 \leq g(1-e_v)-e_f$ and $d_1 > g(1-e_v) - e_f$.

In the interval $0 < d_1 \leq g(1-e_v) - e_f$ the guarantee is always greater than a'', hence, for a production guarantee, $E[\tilde{S}_1]$ is calculated from (2.49), D is equal to d_1/R_f and the expected return on investment is given by

$$\begin{aligned} E[\widetilde{ROE}] \quad = \quad & \frac{1}{A - \frac{d_1}{R_f}}(1-T)\left\{(1-e_v)\left[E[\widetilde{Rev}](1-F_{Rev}(g)) + gF_{Rev}(g) + \sigma^2_{Rev} f_{Rev}(g)\right] - \right. \\ & \left. -d_1 - e_f\right\} + T \qquad (2.102) \end{aligned}$$

Here, $E[\widetilde{ROE}]$ increases as d_1 increases and hence, $E[\widetilde{ROE}]^g_{\max}$ always occurs at the boundary $d_1 = g(1-e_v) - e_f$ (i.e., $g = a''$). In the interval $d_1 > g(1-e_v) - e_f$ the guarantee falls in between b'' and a'', hence, for a production guarantee, $E[\tilde{S}_1]$ is calculated from (2.42), D from (2.27) and the expected return on investment is given by

$$\begin{aligned} E[\widetilde{ROE}] \quad = \quad & \frac{1}{A - D}\left\{(1-T)\left[(1-e_v)E[\widetilde{Rev}] - d_1 - e_f\right](1 - F_{Rev}(a'')) + \right. \\ & \left. + (1-T)(1-e_v)\sigma^2_{Rev} f_{Rev}(a'')\right\} + T(1 - F_{Rev}(a'')) \qquad (2.103) \end{aligned}$$

and $E[\widetilde{ROE}]^s_{\max}$ is determined in a manner similar to the one used in the case where the guarantee is smaller than b''. (See equations (2.96), (2.99), (2.100), and (2.101).)[2]

Note that **an increase in the guarantee amount, provided $g > b''$, results in an increase of $E[\widetilde{ROE}]$.** The reasons for this increase depends on the region g falls in. That is, if g is smaller than a'', it can be seen from (2.103) that as g increases everything remains constant but D which increases, hence $E[\widetilde{ROE}]$ also increases. When g gets greater than a'', $D = d_1/R_f$ is not altered by an increase in g but $E[\tilde{S}_1]$ increases, therefore $E[\widetilde{ROE}]$, given by (2.102), also increases. The same results hold true for minimum-revenue guarantees.

2.6 Value of Guarantee

For debtholders, the value of a guarantee, at a given d_1, can be calculated by subtracting the market value of debt without any guarantee, D, from the market value of debt with a guarantee, D_p or D_m. Similarly, for equityholders, the value of a guarantee, for a given d_1, is determined by the difference between the market value of equity with a guarantee, S_p or S_m, and the market value of equity without any guarantee, S.

A guarantee can be viewed as a put option; that is, at the end of the period, if the revenues are larger than the guarantee, then the option expires worthless. However, if revenues are smaller than the guarantees, then the option is exercised and debtholders receive the guarantee amount from equityholders. Guarantees always enhance the market value of a project[3] because they reduce some of the revenue risks and hence, decrease V_{opt}.[4]

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[2]When $E[\widetilde{ROE}]_{\max} = E[\widetilde{ROE}]^g_{\max}$ (i.e., the maximum expected return on investment occurs at $g = a''$) then the objective of maximizing the equityholders' wealth provides the same ''optimal'' capital structure as the objective of maximizing their returns.

[3]The value of the debt also increases with the presence of project guarantees. The value of equity, however, presents mixed results as the guarantee increases the value of equity for certain values of d_1 and decreases it for large values of d_1.

[4]From last chapter, $V_{\tilde{X}} = D + S + V_{gov} + Vopt$.

Table 2-2: Input parameters for the example project (all \$ values are in millions)

Project		Market	
Variable (1)	Value (2)	Variable (3)	Value (4)
A	\$ 2,200	$E[\tilde{R}_m]$	1.12
$E[\widetilde{Rev}]$	\$ 3,800	σ_m	0.20
σ_{Rev}	\$ 1,500	T	0.35
ρ_{Rev,R_m}	0.70	R_f	1.06
e_f	\$ 500		
e_v	0.20		
b_f	\$ 100		
b_v	0.30		

2.7 Example

This section presents an example to illustrate the concepts developed in previous sections of this chapter. Table 2-2 shows the input parameters necessary for the determination of the debt capacity and the optimal capital structure of a privately-financed project with either a production or a minimum-revenue guarantee and displays the specific values assumed for the parameters in this example.

Table 2-3 contains, for the case of a project with no guarantees, the numerical values of D, S, V, NPV, D/A, $E[\tilde{r}_D]$, *Int*, $E[\tilde{r}_S]$, $E[\widetilde{\text{Roe}}]$, $\partial D/\partial d_1$, and $\partial E[\widetilde{\text{Roe}}]/\partial d_1$ for different d_1 values. This table is similar to Table 1-2. Tables 2-4 and 2-5 contain the same information of Table 2-3 for the case of a project with a \$2,600 production guarantee and with a \$2,600 minimum-revenue guarantee respectively. The maximum guarantee of this project is:

$$g_{\max} = \frac{AR_f + e_f}{1 - e_v} = \$3,540.00 \tag{2.104}$$

For the production guarantee, the market value of debt, D_p, is calculated from (2.24), (2.31), and (2.33) and the market value of equity, S_p, is given from (2.46), and (2.52). For the minimum-revenue guarantee, D_m is given from (2.24), (2.73), and (2.33) and S_m from (2.46), and (2.88).

The present value of the project, V, is the sum of D and S. Of course, this is only valid for $S \geq 0$. The net present value, NPV, is equal to the market value of the project, V, minus the initial project costs, A. The percentage of debt financing used in the project, D/A, is the ratio between the present value of the project's debt (*i.e.*, the amount of money debtholders will provide to the project) and the initial cost of the project.

The effective return on debt, $E[\tilde{r}_D]$, is the expected return for the debtholders, thus, $D(1 + E[\tilde{r}_D])$ is the repayment amount debtholders expect to receive at the end of the period. The promised return on debt, ***Int***, is the interest rate debtholders would charge the owning company in order to lend them D. ***Int*** is calculated as the ratio between d_1 and D, minus one.

The required return on equity, $E[\tilde{r}_S]$, is the return investors would expect to receive if they had invested in an openly-traded asset that presents the same degree of riskness as the concession-financed project (*i.e.*, $\beta_{\text{asset}} = \beta_{\text{project}}$). The expected return on equity investment, $E[\widetilde{\text{Roe}}]$, is the ratio between the expected end-of-period value of the project after all obligations have been paid and the amount infused by investors at the beginning of the period, minus one. The rates of change, $\partial D/\partial d_1$ and $\partial E[\widetilde{\text{Roe}}]/\partial d_1$, are calculated from the formulation developed in Sections 2.4 and 2.5.

Figure 2-5 displays D, S, and V, as a function of d_1 for the following cases: no guarantee, production guarantee equal to \$2,600 and minimum-revenue guarante equal to \$2,600. Acording to (2.10) (or 2.56), the value of the debt is the amount of money debtholders expect to receive at the end of the period minus the systematic risk premium on the project's debt (*i.e.*, the amount lenders charge to buy part of the project's systematic operating risk premium from the owning company), divided by R_f.

At $d_1 = d_1^c$, the market value of the debtholders' holdings reaches a maximum, therefore D^c is the maximum amount of money the owning company can borrow from debtholders (*i.e.*, the debt capacity of the project). This maximum amount varies according to the type of guarantee the project has. From Fig. 2-5 it can be seen that no-guarantee provides the lowest debt capacity amount while the minimum-revenue guarantee provides the largest one. Guarantees increase debt capacity because they decrease the risk profile of the loan enabling promoters to borrow more money while promising to pay a fixed amount. In this example, the minimum-revenue guarantee provides a greater debt capacity than the production guarantee because the condition

Promised debt amount d_1 (1)	Market value of debt D (2)	Market value of equity S (3)	Market value of project V (4)	Net Present Value *NPV* (5)	Debt financing D/A (6)	Effective return on debt $E[\tilde{r}_D]$ (7)	Promised return on debt *Int* (8)	Required return on equity $E[\tilde{r}_S]$ (9)	Return on equity investment $E[\widetilde{Roe}]$ (10)	$\partial D/\partial d_1$ (11)	$\partial E[\widetilde{ROE}]/\partial d_1$ (12)
0	0.00	2,117.70	2,117.70	-82.30	0.000	0.060	0.060	0.139	0.097	0.914	9.91E-06
171	155.78	1,959.00	2,114.78	-85.22	0.071	0.073	0.100	0.146	0.098	0.900	6.17E-06
343	308.34	1,801.70	2,110.03	-89.97	0.140	0.076	0.111	0.154	0.099	0.881	1.37E-07
514	457.11	1,646.40	2,103.51	-96.49	0.208	0.080	0.124	0.162	0.098	0.855	-8.92E-06
685	600.99	1,494.03	2,095.02	-104.98	0.273	0.084	0.140	0.172	0.095	0.823	-2.20E-05
857	738.77	1,345.65	2,084.42	-115.58	0.336	0.089	0.159	0.184	0.090	0.784	-4.04E-05
1,028	869.11	1,202.43	2,071.54	-128.46	0.395	0.094	0.183	0.197	0.081	0.737	-6.55E-05
1,199	990.65	1,065.61	2,056.26	-143.74	0.450	0.100	0.210	0.211	0.067	0.681	-9.88E-05
1,370	1,102.09	936.34	2,038.43	-161.57	0.501	0.107	0.243	0.227	0.047	0.619	-1.42E-04
1,542	1,202.22	815.69	2,017.91	-182.09	0.546	0.114	0.282	0.245	0.018	0.550	-1.95E-04
1,580	1,222.97	789.98	2,012.94	-187.06	0.556	0.116	0.292	0.250	0.010	0.533	-2.08E-04
1,581	1,223.50	789.31	2,012.81	-187.19	0.556	0.116	0.292	0.250	0.010	0.533	-2.08E-04
1,713	1,290.10	704.46	1,994.57	-205.43	0.586	0.122	0.328	0.265	-0.020	0.476	-2.57E-04
1,884	1,365.07	603.21	1,968.27	-231.73	0.620	0.130	0.380	0.287	-0.070	0.399	-3.25E-04
2,056	1,426.82	512.14	1,938.96	-261.04	0.649	0.138	0.441	0.311	-0.132	0.322	-3.94E-04
2,227	1,475.49	431.19	1,906.68	-293.32	0.671	0.147	0.509	0.337	-0.204	0.247	-4.54E-04
2,398	1,511.63	359.98	1,871.61	-328.39	0.687	0.155	0.586	0.365	-0.286	0.176	-4.99E-04
2,570	1,536.17	297.94	1,834.12	-365.88	0.698	0.163	0.673	0.395	-0.374	0.112	-5.20E-04
2,741	1,550.41	244.37	1,794.78	-405.22	0.705	0.171	0.768	0.427	-0.463	0.056	-5.15E-04
2,912	1,555.86	198.49	1,754.35	-445.65	0.707	0.178	0.872	0.463	-0.549	0.009	-4.88E-04
3,083	1,554.21	159.52	1,713.74	-486.26	0.706	0.184	0.984	0.501	-0.629	-0.027	-4.43E-04
3,255	1,547.19	126.71	1,673.90	-526.10	0.703	0.189	1.104	0.543	-0.701	-0.053	-3.89E-04
3,426	1,536.44	99.37	1,635.80	-564.20	0.698	0.193	1.230	0.589	-0.762	-0.071	-3.31E-04

Table 2-3: Example Results for the Case of No Guarantees (all $ in millions)

Promised debt amount d_1 (1)	Market value of debt D (2)	Market value of equity S (3)	Market value of project V (4)	Net Present Value *NPV* (5)	Debt financing D/A (6)	Effective return on debt $E[\tilde{r}_D]$ (7)	Promised return on debt *Int* (8)	Required return on equity $E[\tilde{r}_S]$ (9)	Return on equity investment $E[\widetilde{Roe}]$ (10)	$\partial D/\partial d_1$ (11)	$\partial E[\widetilde{ROE}]/\partial d_1$ (12)
0	0.00	2,250.63	2,250.63	50.63	0.000	0.060	0.060	0.117	0.143	0.943	4.46E-05
171	161.60	2,092.22	2,253.83	53.83	0.073	0.060	0.060	0.122	0.151	0.943	5.20E-05
343	323.21	1,933.82	2,257.03	57.03	0.147	0.060	0.060	0.127	0.161	0.943	6.13E-05
514	484.81	1,775.42	2,260.23	60.23	0.220	0.060	0.060	0.133	0.172	0.943	7.34E-05
685	646.42	1,617.02	2,263.43	63.43	0.294	0.060	0.060	0.140	0.186	0.943	8.95E-05
857	808.02	1,458.62	2,266.63	66.63	0.367	0.060	0.060	0.149	0.203	0.943	1.11E-04
1,028	969.62	1,300.21	2,269.84	69.84	0.441	0.060	0.060	0.159	0.225	0.943	1.43E-04
1,199	1,131.23	1,141.81	2,273.04	73.04	0.514	0.060	0.060	0.173	0.253	0.943	1.89E-04
1,370	1,292.83	983.41	2,276.24	76.24	0.588	0.060	0.060	0.191	0.291	0.943	2.62E-04
1,542	1,454.43	825.01	2,279.44	79.44	0.661	0.060	0.060	0.216	0.346	0.943	3.89E-04
1,580	1,490.57	789.59	2,280.16	80.16	0.678	0.060	0.060	0.224	0.362	0.943	4.29E-04
1,581	1,343.43	760.52	2,103.96	-96.04	0.611	0.086	0.177	0.254	0.113	0.533	-1.61E-04
1,713	1,410.04	677.19	2,087.23	-112.77	0.641	0.093	0.215	0.270	0.088	0.476	-2.16E-04
1,884	1,485.00	578.04	2,063.04	-136.96	0.675	0.102	0.269	0.292	0.044	0.399	-2.99E-04
2,056	1,546.76	489.19	2,035.95	-164.05	0.703	0.111	0.329	0.316	-0.014	0.322	-3.88E-04
2,227	1,595.43	410.52	2,005.95	-194.05	0.725	0.119	0.396	0.342	-0.089	0.247	-4.75E-04
2,398	1,631.57	341.61	1,973.18	-226.82	0.742	0.128	0.470	0.371	-0.176	0.176	-5.45E-04
2,570	1,656.11	281.85	1,937.96	-262.04	0.753	0.136	0.552	0.401	-0.274	0.112	-5.88E-04
2,741	1,670.34	230.47	1,900.82	-299.18	0.759	0.143	0.641	0.435	-0.376	0.056	-5.97E-04
2,912	1,675.80	186.67	1,862.47	-337.53	0.762	0.150	0.738	0.470	-0.476	0.009	-5.73E-04
3,083	1,674.15	149.63	1,823.78	-376.22	0.761	0.155	0.842	0.509	-0.571	-0.027	-5.23E-04
3,255	1,667.12	118.57	1,785.70	-414.30	0.758	0.160	0.952	0.551	-0.655	-0.053	-4.59E-04
3,426	1,656.37	92.78	1,749.15	-450.85	0.753	0.163	1.068	0.597	-0.727	-0.071	-3.89E-04

Table 2-4: Example Results for the Case of Production Guarantee = $ 2,600 (all $ in millions)

Promised debt amount d_1 (1)	Market value of debt D (2)	Market value of equity S (3)	Market value of project V (4)	Net Present Value *NPV* (5)	Debt financing D/A (6)	Effective return on debt $E[\tilde{r}_D]$ (7)	Promised return on debt *Int* (8)	Required return on equity $E[\tilde{r}_S]$ (9)	Return on equity investment $E[\widetilde{Roe}]$ (10)	$\partial D/\partial d_1$ (11)	$\partial E[\widetilde{ROE}]/\partial d_1$ (12)
0	0.00	2,280.92	2,280.92	80.92	0.000	0.060	0.060	0.113	0.154	0.943	4.92E-05
171	161.60	2,122.52	2,284.13	84.13	0.073	0.060	0.060	0.117	0.163	0.943	5.73E-05
343	323.21	1,964.12	2,287.33	87.33	0.147	0.060	0.060	0.121	0.173	0.943	6.76E-05
514	484.81	1,805.72	2,290.53	90.53	0.220	0.060	0.060	0.127	0.186	0.943	8.09E-05
685	646.42	1,647.32	2,293.73	93.73	0.294	0.060	0.060	0.133	0.201	0.943	9.87E-05
857	808.02	1,488.91	2,296.93	96.93	0.367	0.060	0.060	0.141	0.220	0.943	1.23E-04
1,028	969.62	1,330.51	2,300.13	100.13	0.441	0.060	0.060	0.151	0.244	0.943	1.57E-04
1,199	1,131.23	1,172.11	2,303.34	103.34	0.514	0.060	0.060	0.163	0.275	0.943	2.08E-04
1,370	1,292.83	1,013.71	2,306.54	106.54	0.588	0.060	0.060	0.179	0.317	0.943	2.89E-04
1,542	1,454.43	855.30	2,309.74	109.74	0.661	0.060	0.060	0.201	0.378	0.943	4.28E-04
1,580	1,490.57	819.89	2,310.45	110.45	0.678	0.060	0.060	0.207	0.395	0.943	4.73E-04
1,581	1,379.84	751.78	2,131.62	-68.38	0.627	0.076	0.146	0.255	0.150	0.533	-1.41E-04
1,713	1,446.45	668.92	2,115.36	-84.64	0.657	0.083	0.184	0.271	0.128	0.476	-1.97E-04
1,884	1,521.41	570.40	2,091.81	-108.19	0.692	0.092	0.239	0.293	0.087	0.399	-2.84E-04
2,056	1,583.16	482.23	2,065.39	-134.61	0.720	0.101	0.298	0.318	0.030	0.322	-3.82E-04
2,227	1,631.83	404.24	2,036.08	-163.92	0.742	0.110	0.365	0.344	-0.044	0.247	-4.78E-04
2,398	1,667.97	336.04	2,004.01	-195.99	0.758	0.118	0.438	0.373	-0.133	0.176	-5.60E-04
2,570	1,692.52	276.96	1,969.48	-230.52	0.769	0.126	0.518	0.404	-0.234	0.112	-6.13E-04
2,741	1,706.75	226.26	1,933.00	-267.00	0.776	0.133	0.606	0.437	-0.341	0.056	-6.29E-04
2,912	1,712.20	183.09	1,895.29	-304.71	0.778	0.140	0.701	0.473	-0.447	0.009	-6.07E-04
3,083	1,710.56	146.63	1,857.19	-342.81	0.778	0.145	0.803	0.512	-0.547	-0.027	-5.56E-04
3,255	1,703.53	116.10	1,819.63	-380.37	0.774	0.150	0.911	0.554	-0.637	-0.053	-4.87E-04
3,426	1,692.78	90.78	1,783.55	-416.45	0.769	0.153	1.024	0.600	-0.714	-0.071	-4.12E-04

Table 2-5: Example Results for the Case of Minimum-Revenue Guarantee = $ 2,600 (all $ in millions)

$g \geq E[\widetilde{Rev}] - \lambda\ \text{Cov}(\widetilde{Rev}, \tilde{R}_m)/(1-b_v) - \sigma^2_{Rev} f_{Rev}(g)/F_{Rev}(g)$ is satisfied.[5]

Figure 2-5 also shows discontinuity on the curves related to guarantees. This occurs because as d_1 increases the guarantee changes from being larger than a'' to being smaller than a''. Therefore, it is necessary to use two functions to express the market values of debt and equity of a project with either a production or a minimum-revenue guarantee and the determination of project debt capacity requires the comparison of the maximum point of both of these functions. (See table 2-1.) In this example, a \$2,600 production guarantee yields: $D^g_{\max} = \$1,491$ and $D^s_{\max} = \$1,676 < A$ therefore $D^c = D^s_{\max} = \$1,676$. A \$2,600 minimum-revenue guarantee gives: $D^g_{\max} = \$1,491$, $D^s_{\max} = \$1,712 < A$, and $D^c = D^s_{\max} = \$1,712$.

Figure 2-6 displays $E[\widetilde{Roe}]$ as a function of d_1 for the cases of: no guarantee, production guarantee equal to \$2,600 and minimum-revenue guarante equal to \$2,600. The minimum-revenue-guarantee scenario produces the largest expected return to investors while the no-guarantee scenario produces the smallest. Guarantees increase $E[\widetilde{Roe}]$ because they enable promoters to borrow more money from lenders while promising to pay a fixed amount, and hence require a lower equity investment.

Figure 2-6 also shows discontinuity on the curves related to guarantees. This occurs because as d_1 increases the guarantee changes from being larger than a'' to being smaller than a''. Therefore, in order to determine the optimal capital structure for the project (*i.e.,* $E[\widetilde{Roe}]_{\max}$), it is necessary to use a procedure similar to the one used to determine the project debt capacity. In this example, a \$2,600 production guarantee yields: $E[\widetilde{Roe}]^g_{\max} = 0.362$ and $E[\widetilde{Roe}]^s_{\max} = -0.476$ therefore $E[\widetilde{Roe}]_{\max} = E[\widetilde{Roe}]^g_{\max} = 0.362$. A \$2,600 minimum-revenue guarantee gives: $E[\widetilde{Roe}]^g_{\max} = 0.395$, $E[\widetilde{Roe}]^s_{\max} = -0.447$, and $E[\widetilde{Roe}]_{\max} = E[\widetilde{Roe}]^g_{\max} = 0.395$.

As $E[\widetilde{Roe}]_{\max} = E[\widetilde{Roe}]^g_{\max}$ then the objective of maximizing the project's NPV provides the same ‘‘optimal capital structure’’ as the objective of maximizing the equityholders's returns. Moreover, in this example, the presence of a guarantee is sufficient to turn the negative-NPV project (*e.g.,* -82.30) into a positive-NPV one. For the \$2,600 production and minimum-revenue guarantees, at $D^{Roe} = D^{NPV}$, the project's NPV are 80.16 and 110.45, respectively.

Figure 2-7 displays $E[\tilde{r}_D]$, *Int*, and $E[\tilde{r}_S]$ as a function of d_1 for the cases of: no guarantee,

[5] If $d_1^c \leq g(1-e_v) - ef$ (i.e., guarantee $\geq a''$), both types of guarantees provide the same collateral to the loan, and hence, produce equal present values of debt for the project.

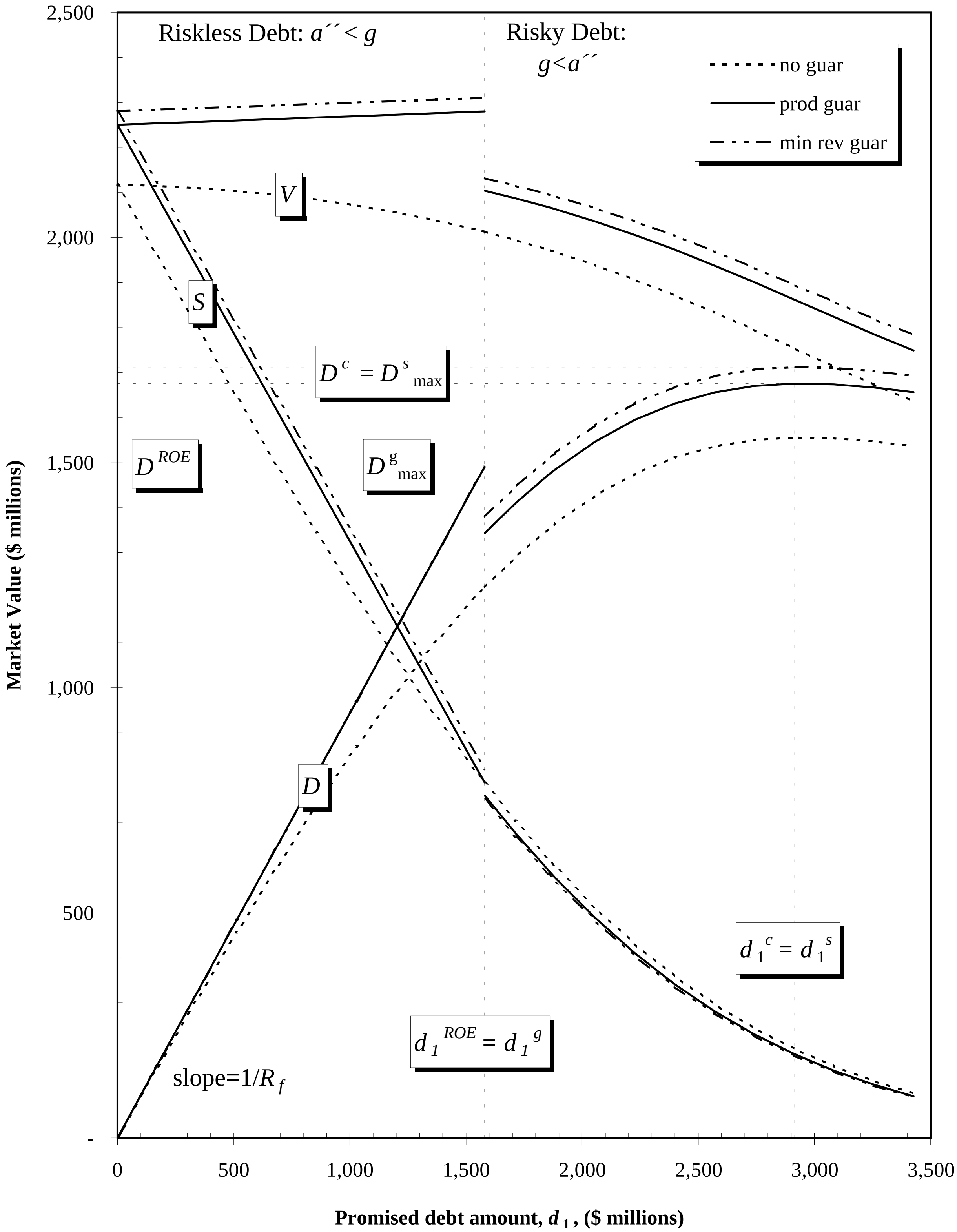

Figure 2-5: Present (market) values of V, D, and S as a function of d_1

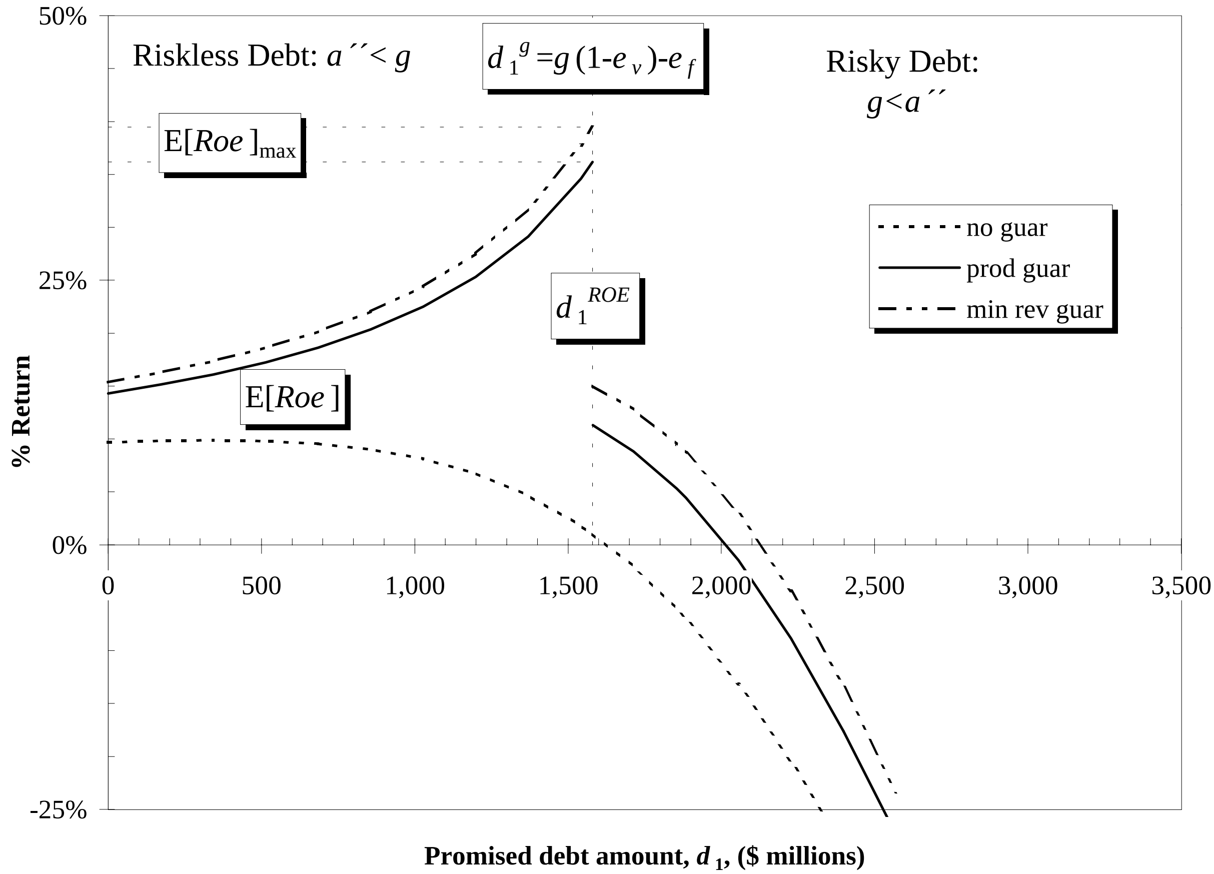

Figure 2-6: Expected value of the return to investors, $E[\widetilde{Roe}]$, as a function of d_1

production guarantee equal to $2,600, and minimum-revenue guarante equal to $2,600. Lenders perceive projects with guarantees as less risky when compared to projects without guarantees. Thus, in the presence of guarantees, they lend money at a lower interest rate and also expect a lower return on debt than the base case of no guarantees. As in Figures 2-5 and 2-6, Figure 2-7 also shows a discontinuity on the curves related to guarantees. Note that $E[\tilde{r}_D] = \mathit{Int} = R_f$ for promised debt amounts to the left of the discontinuity. This is because, at these borrowing levels ($d_1 < g(1 - e_v) - e_f$), the guarantee is sufficient to provide debt repayment, and thus, the debt is riskless.

Figure 2-8 displays D^c/A (*i.e.*, the project debt capacity as a percentage of the initial project cost) at different guarantee levels. As expected, it shows that the debt capacity is directly proportional to the amount of guarantees. It also shows that debt capacity presents two different

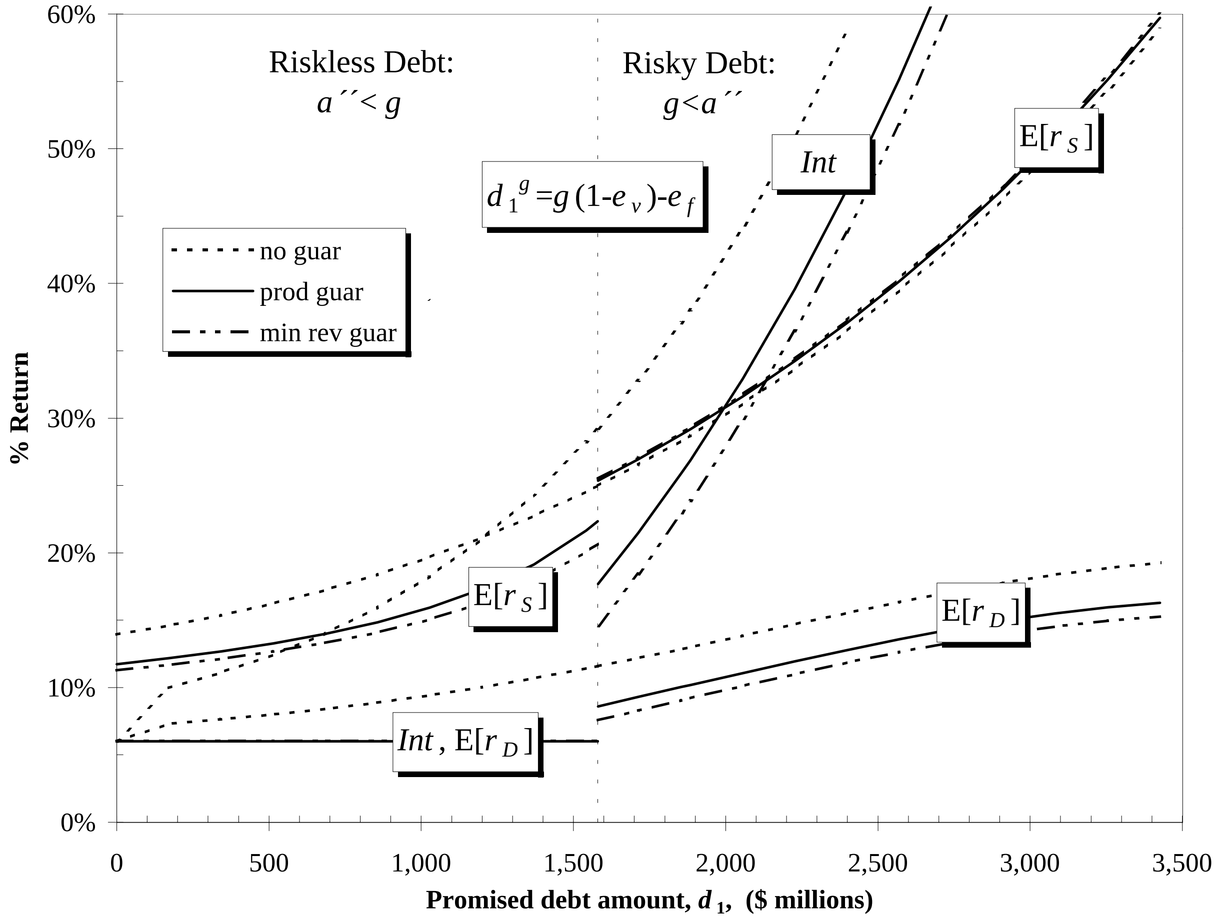

Figure 2-7: $E[\tilde{r}_D]$, *Int*, and $E[\tilde{r}_S]$, as a function of d_1

behaviors. In region ''AB,'' $D^c = D^s_{\max}$,[6] that is, debt capacity occurs within the region where $g < a''$ (*i.e.*, the debt is risky). In region ''BC,'' $D^c = D^g_{\max}$, that is, debt capacity occurs at the region where guarantee $\geq a''$ (*i.e.*, the debt is riskless). In this case, $D^c = D^g_{\max}$ occurs at the extreme right, i.e., where d_1 is such that it corresponds to a'' (*i.e.*, where the company promises to repay the maximum amount fully covered by the guarantees). Point ''A'' represents the case of no guarantees, point ''B'' indicates the case where $D^c = D^s_{\max} = D^g_{\max}$, and point ''C'' represents the case where the guarantee amount is sufficient to secure 100% debt financing (*i.e.*, $g = g_{\max}$).

[6] See Table 2-1.

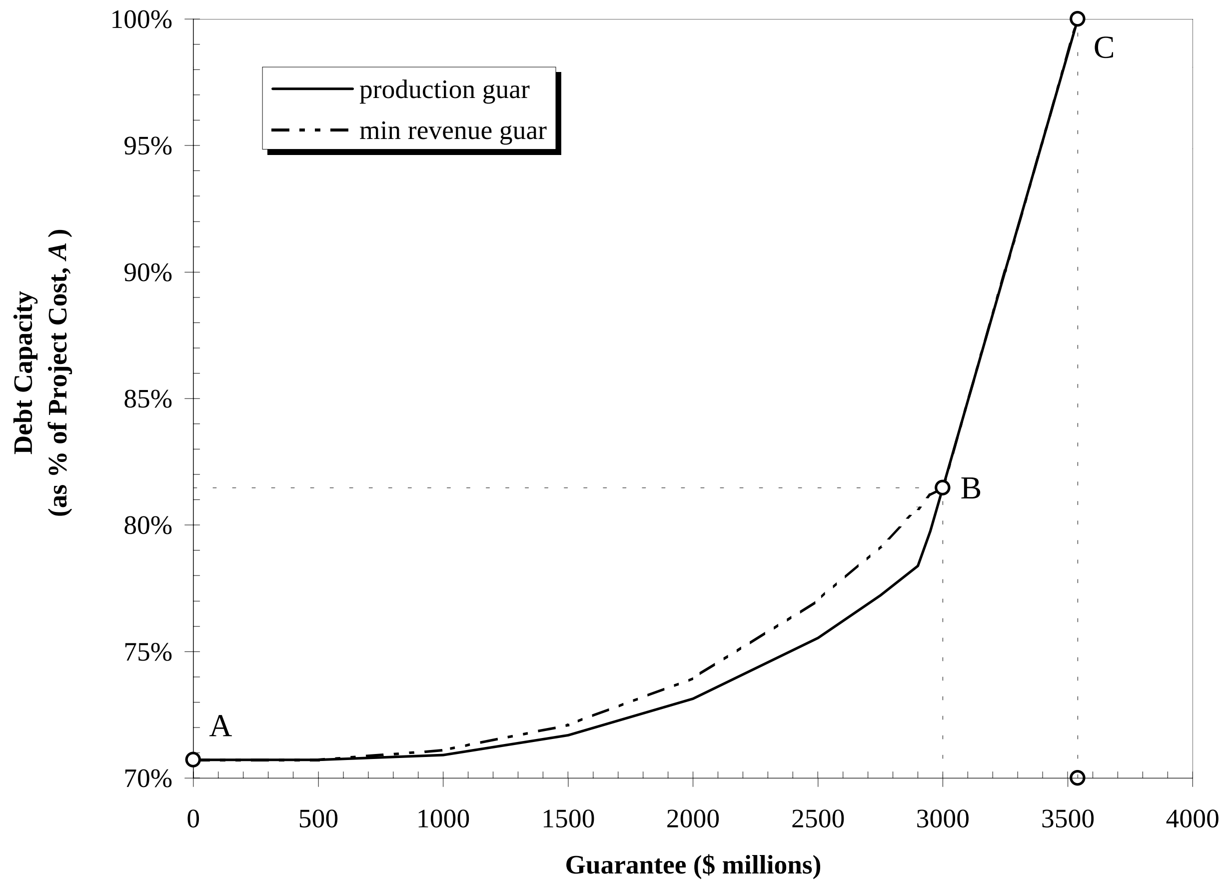

Figure 2-8: Effect of guarantees on project debt capacity

2.8 Summary

We have shown in this chapter that the presence of a guarantee greater than b'' increases the debt capacity of a project. This is because the project becomes less risky and lenders are willing to finance it at a lower interest rate than they would if there are no guarantees, that is, for the same promised amount d_1 promoters would be able to borrow D_p or D_m which is greater than D without guarantees. If the guarantee is equal or greater than a'', then the debt is riskless and the possibility of bankruptcy does not exist.

We have also shown that the expected return on investment for the promoters increases with the presence of guarantees. If the guarantee is smaller than a'', the increase in the expected return is due to the necessity to infuse less capital (*i.e.*, the amount that can be borrowed increases) while the expected value of equity at the end of the period remains constant. If the guarantee is equal or greater than a'', the increase in the expected return occurs because the expected value of equity at the end of the period increases while the amount that can be borrowed remains constant (*i.e.*, $D = d_1/R_f$). If $E[\widetilde{ROE}]_{\max} = E[\widetilde{ROE}]^g_{\max}$ (*i.e.*, the maximum expected return on investment occurs at $g = a''$) then the objective of maximizing the project's net present value provides the same ‘‘optimal’’ capital structure as the objective of maximizing the equityholders' returns. If $E[\widetilde{ROE}]_{\max} = E[\widetilde{ROE}]^g_{\max}$ and $D^c = D^g_{\max}$, the maximum expected return on the equityholders' investment as well as the maximum project's NPV occur at project debt capacity, otherwise the ‘‘optimal’’ capital structure always requires less borrowing than the available debt capacity.

Appendix A

Derivation of Auxiliary Mathematical Relationships

A.1 Basic Formulas for Normal Random Variables

A.1.1 Partial moments of Normally distributed random variables

The first partial moment of a Normal distribution is given by

$$E_a^b[\tilde{X}] = \int_a^b \tilde{X} f_X(\tilde{X}) d\tilde{X} \tag{A.1}$$

$$= E[\tilde{X}](F_X(b) - F_X(a)) + \sigma_X^2 (f_X(a) - f_X(b)) \tag{A.2}$$

Proof:

$$\int_a^b \tilde{X} f_X(\tilde{X}) d\tilde{X} = \int_a^b \frac{\tilde{X}}{\sqrt{2\pi}\sigma_X} e^{-\frac{1}{2}\left(\frac{\tilde{X}-m_X}{\sigma_X}\right)^2} d\tilde{X} \tag{A.3}$$

Substituting $\tilde{X}$ by $m_X + \tilde{U}\sigma_X$ and $d\tilde{X}$ by $\sigma_X d\tilde{U}$ in the above equation we have:

$$\begin{aligned}
\int_a^b \tilde{X} f_X(\tilde{X}) d\tilde{X} &= \int_{\frac{a-m_X}{\sigma_X}}^{\frac{b-m_X}{\sigma_X}} \frac{m_X + \tilde{U}\sigma_X}{\sqrt{2\pi}} e^{-\frac{1}{2}\tilde{U}^2} d\tilde{U} \\
&= \frac{m_X}{\sqrt{2\pi}} \int_{\frac{a-m_X}{\sigma_X}}^{\frac{b-m_X}{\sigma_X}} e^{-\frac{1}{2}\tilde{U}^2} d\tilde{U} + \frac{\sigma_X}{\sqrt{2\pi}} \int_{\frac{a-m_X}{\sigma_X}}^{\frac{b-m_X}{\sigma_X}} \tilde{U} e^{-\frac{1}{2}\tilde{U}^2} d\tilde{U} \\
&= m_X \left[F_U\left(\frac{b-m_X}{\sigma_X}\right) - F_U\left(\frac{a-m_X}{\sigma_X}\right) \right] + \\
&\quad + \sigma_X^2 \left[\frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{1}{2}\left(\frac{a-m_X}{\sigma_X}\right)^2} - \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{1}{2}\left(\frac{b-m_X}{\sigma_X}\right)^2} \right] \\
&= m_X (F_X(b) - F_X(a)) + \sigma_X^2 (f_X(a) - f_X(b))
\end{aligned} \tag{A.4}$$

The second partial moment of a Normal distribution is given by

$$\begin{aligned} E_a^b[\tilde{X}^2] &= \int_a^b \tilde{X}^2 f_X(\tilde{X}) d\tilde{X} \\ &= (E[\tilde{X}] + \sigma_X^2)(F_X(b) - F_X(a)) + \\ &\quad + \sigma_X^2 \{E[\tilde{X}](f_X(a) - f_X(b)) + a f_X(a) - b f_X(b)\} \end{aligned} \tag{A.5}$$

The proof of equation (A.5) follows a reasoning similar to the one used to prove equation (A.2). Alternatively, Winkler *et.al.* (1972, pp. 294) provide an equation that can be used to calculate the partial n^{th} moment of a Normal distribution,

$$\begin{aligned} E_{-\infty}^b[\tilde{X}^n] &= \int_{-\infty}^b \tilde{X}^n f_X(\tilde{X}) d\tilde{X} \\ &= -\sigma_X^2 b^{n-1} f_N(b) + (n-1)\sigma_X^2 E_{-\infty}^b[\tilde{X}^{n-2}] + E[\tilde{X}]\, E_{-\infty}^b[\tilde{X}^{n-1}] \end{aligned} \tag{A.6}$$

For $n = 2$:

$$\begin{aligned} E_{-\infty}^b[\tilde{X}^2] &= \int_{-\infty}^b \tilde{X}^2 f_X(\tilde{X})\, d\tilde{X} \\ &= -\sigma_X^2 b f_X(b) + \sigma_X^2 E_{-\infty}^b[\tilde{X}^0] + E[\tilde{X}]\, E_{-\infty}^b[\tilde{X}^1] \\ &= -\sigma_X^2 b f_X(b) + \sigma_X^2 F_X(b) + E[\tilde{X}] \int_{-\infty}^b \tilde{X} f_X(\tilde{X})\, d\tilde{X} \end{aligned} \tag{A.7}$$

Therefore,

$$\begin{aligned} E_a^b[\tilde{X}^2] &= \sigma_X^2\,(a f_X(a) - b f_X(b)) + \sigma_X^2\,(F_X(b) - F_X(a)) + \\ &\quad + E[\tilde{X}] \int_a^b \tilde{X} f_X(\tilde{X})\, d\tilde{X} \end{aligned} \tag{A.8}$$

$$\begin{aligned} &= (E[\tilde{X}] + \sigma_X^2)(F_X(b) - F_X(a)) + \\ &\quad + \sigma_X^2 \{E[\tilde{X}](f_X(a) - f_X(b)) + a f_X(a) - b f_X(b)\} \end{aligned} \tag{A.9}$$

A.1.2 Conditional expected values for jointly Normally distributed random variables

According to Benjamin and Cornell (1970, pp. 421) iff $\tilde{X}$ and $\tilde{R}_m$ are jointly Normally distributed, then

$$E[\tilde{R}_m | \tilde{X} = x] = \int_{-\infty}^{\infty} \tilde{R}_m\, f_{R_m|X}(\tilde{R}_m | \tilde{X} = x)\, d\tilde{R}_m$$

$$\begin{aligned} &= E[\tilde{R}_m] + \rho_{R_m,X} \frac{\sigma_{R_m}}{\sigma_X} \left(\tilde{X} - E[\tilde{X}]\right) \\ &= E[\tilde{R}_m] + \frac{\text{Cov}(\tilde{X}, \tilde{R}_m)}{\sigma_X^2} \left(\tilde{X} - E[\tilde{X}]\right) \end{aligned} \tag{A.10}$$

A.2 Determination of Expected Values

Given that δ_b and δ_q are Bernoulli random variables with a certain probability of success and that $\tilde{X}$ and $\tilde{R}_m$ are Normally distributed random variables then,

$$E[\delta_b] = P[\text{ Success}] = F_X(d_1) \tag{A.11}$$

$$E[\delta_q] = 1 - F_X\left(\frac{b_f}{1 - b_v}\right) = 1 - F_X(b') \tag{A.12}$$

$$E[\delta_b \delta_q] = F_X(d_1) - F_X\left(\frac{b_f}{1 - b_v}\right) = F_X(d_1) - F_X(b') \tag{A.13}$$

$$E[\delta_b \delta_q \tilde{X}] = \int_{-\infty}^{\infty} \delta_b \delta_q \tilde{X} f_X(\tilde{X}) d\tilde{X} = \int_{b'}^{d_1} \tilde{X} f_X(\tilde{X}) d\tilde{X} \tag{A.14}$$

The above equation is similar to (A.1). Therefore,

$$E[\delta_b \delta_q \tilde{X}] = E[\tilde{X}] \left(F_X(d_1) - F_X(b')\right) + \sigma_X^2 \left(f_X(b') - f_X(d_1)\right) \tag{A.15}$$

$$\begin{aligned} E[\delta_b \delta_q \tilde{R}_m] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta_b \delta_q \tilde{R}_m \, f_{X,R_m}(\tilde{X}, \tilde{R}_m) \, d\tilde{R}_m \, d\tilde{X} \\ &= \int_{b'}^{d_1} \delta_b \delta_q \, f_X(\tilde{X}) \int_{-\infty}^{\infty} \tilde{R}_m \, f_{R_m|X}(\tilde{R}_m | \tilde{X} = x) \, d\tilde{R}_m \, d\tilde{X} \end{aligned} \tag{A.16}$$

Substituting equation (A.10) into equation (A.16) and from the definitions given above:

$$\begin{aligned} E[\delta_b \delta_q \tilde{R}_m] &= E[\tilde{R}_m] \, E[\delta_b \delta_q] + \frac{\text{Cov}(\tilde{X}, \tilde{R}_m)}{\sigma_X^2} \Big\{ E[\tilde{X}] \left(F_X(d_1) - F_X(b')\right) + \\ &\quad + \sigma_X^2 \left(f_X(b') - f_X(d_1)\right) - E[\tilde{X}] \left(F_X(d_1) - F_X(b')\right) \Big\} \end{aligned} \tag{A.17}$$

$$
\begin{aligned}
E[\delta_b \delta_q \tilde{X} \tilde{R}_m] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta_b \delta_q \tilde{X} \tilde{R}_m f_{X,R_m}(\tilde{X}, \tilde{R}_m)\, d\tilde{R}_m \, d\tilde{X} \\
&= \int_{b'}^{d_1} \delta_b \delta_q \tilde{X} f_X(\tilde{X}) \int_{-\infty}^{\infty} \tilde{R}_m f_{R_m|X}(\tilde{R}_m | \tilde{X} = x)\, d\tilde{R}_m \, d\tilde{X} \qquad \text{(A.18)}
\end{aligned}
$$

Substituting equations (A.10), (A.5) and (A.2) into equation (A.18) gives:

$$
\begin{aligned}
E[\delta_b \delta_q \tilde{X} \tilde{R}_m] &= E[\tilde{R}_m] \left\{ E[\tilde{X}] \left(F_X(d_1) - F_X(b') \right) + \sigma_X^2 \left(f_X(b') - f_X(d_1) \right) \right\} + \\
&\quad + \frac{\text{Cov}(\tilde{X}, \tilde{R}_m)}{\sigma_X^2} \left\{ \sigma_X^2 \left(b' f_X(b') - d_1 f_X(d_1) \right) + \right. \\
&\quad \left. + \sigma_X^2 \left(F_X(d_1) - F_X(b') \right) \right\} \qquad \text{(A.19)}
\end{aligned}
$$

A.3 Determination of Covariances

$$
\text{Cov}(\delta_b \delta_q, \tilde{R}_m) = E[\delta_b \delta_q \tilde{R}_m] - E[\delta_b \delta_q]\, E[\tilde{R}_m] \qquad \text{(A.20)}
$$

Substituting equations (A.13) and (A.17) into equation (A.20) gives:

$$
\begin{aligned}
\text{Cov}(\delta_b \delta_q, \tilde{R}_m) &= \frac{\text{Cov}(\tilde{X}, \tilde{R}_m)}{\sigma_X^2} \sigma_X^2 \left(f_X(b') - f_X(d_1) \right) + E[\tilde{R}_m]\, E[\delta_b \delta_q] - E[\tilde{R}_m]\, E[\delta_b \delta_q] \\
&= \left(f_X(b') - f_X(d_1) \right) \ \text{Cov}(\tilde{X}, \tilde{R}_m) \qquad \text{(A.21)}
\end{aligned}
$$

$$
\text{Cov}(\delta_b \delta_q \tilde{X}, \tilde{R}_m) = E[\delta_b \delta_q \tilde{X} \tilde{R}_m] - E[\delta_b \delta_q \tilde{X}]\, E[\tilde{R}_m] \qquad \text{(A.22)}
$$

Substituting equations (A.17) and (A.19) into equation (A.22) gives:

$$
\begin{aligned}
\text{Cov}(\delta_b \delta_q \tilde{X}, \tilde{R}_m) &= E[\tilde{R}_m] \left\{ E[\tilde{X}] \left(F_X(d_1) - F_X(b') \right) + \sigma_X^2 \left(f_X(b') - f_X(d_1) \right) \right\} + \\
&\quad + \text{Cov}(\tilde{X}, \tilde{R}_m) \left\{ b' f_X(b') - d_1 f_X(d_1) + F_X(d_1) - F_X(b') \right\} - \\
&\quad - E[\tilde{R}_m] \left\{ E[\tilde{X}] \left(F_X(d_1) - F_X(b') \right) + \sigma_X^2 \left(f_X(b') - f_X(d_1) \right) \right\} \\
&= \left\{ F_X(d_1) - F_X(b') + \left(b' f_X(b') - d_1 f_X(d_1) \right) \right\} \ \text{Cov}(\tilde{X}, \tilde{R}_m) \qquad \text{(A.23)}
\end{aligned}
$$

Appendix B

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